A Representation of a Vector Superfield and its Gauge Structure

Dissertation
zur
Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der
Mathematisch-naturwissenschaftlichen Fakultät
der
Universität Zürich
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Genf (GE)

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Zürich 2005

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Abstract

Following Epstein and Glaser one can construct the S-matrix perturbatively using the so-called causal splitting method. This has been proven a successful approach in quantum electrodynamics and in standard electroweak theory. It seems to be reasonable to follow the same ways to construct a supersymmetric quantum field theory.

A fully quantum construction of a vector superfield is given. This field is constructed without referring to a classical field or langrangian defined on a superspace. Instead the superfield arises from a "sandwiching" formula using supersymmetry "generators". This formula is then very similar to the usual treatment of time or space translations in ordinary quantum field theory. The aim of this work is to find a gauge structure defined by a gauge charge operator which factorizes the initial Hilbert space into a physical subspace. Unphysical fields -ghost fields- are needed to obtain a consistent and complete description of the gauge structure. Of course, these ghost fields also turn out to be the components of a super(ghost)field.

The existence and construction of this gauge charge is the main and last result of this work. As in the ordinary case one can then use this operator to construct perturbatively, order by order, a gauge invariant S-matrix for a supersymmetric theory.

Zusammenfassung

In dieser Arbeit wird eine Darstellung für ein Vektor Super-Multiplet gegeben. Dazu wird die perturbative Eichstruktur dieser Darstellung untersucht.
Im ersten Kapitel wird ganz allgemein erklärt, was man unter einem Quantenfeld versteht und es werden einige freie Felder explizit errechnet.
Im zweiten Kapitel wird beschrieben, wie man physikalisch relevante Situationen in einer Beschreibung durch freie Quantenfeldern beschreiben kann. Die S-Matrix wird perturbativ aufgebaut und man beschreibt die Wichtigkeit der Eichstruktur in diesem Falle.
Kapitel Drei beschäftigt sich mit dem Konzept der physikalischen Symmetrien, und der Erweiterung dieses Konzepts zur Supersymmetrie. Es wird auch gezeigt, wie man zur Super-algebra kommt.
Das Kapitel vier beschäftigt sich dann mit der Verwirklichung von chiralen Superfeldern und Geistsuperfeldern, sowie der von einem Vektorsuperfeld. Bei diesem kommt ein spin-3/2 Feld zum Vorschein.
Die Super-Eichstruktur der Superfelder wird im Kapitel fünf behandelt. Die Eichladung wird explizit berechnet und dargestellt.
Das sechste Kapitel zeigt eine Vereinfachte Darstellung des Vektor-Superfelds.
Schliesslich werden im Appendix mehrere Rechnungen gezeigt und durchgeführt.

Acknowledgements

I would like to thank Professor Günter Scharf, my supervisor, for his patience during this research. I am also thankful to Professor Daniel Wyler for his humanity through the early years of chaos and confusion.

Professor Jürg Fröhlich gave me the opportunity to increase substantially my knowing in theoretical physics during a number of personal interviews. He gave me the references for many papers in axiomatical field theory exposed in chapter two. The privilege of meeting him was an immense pleasure.

Professor Hervé Kunz generously invited me at the Swiss Federal Institute of Technology in Lausanne for finishing my writing and gave me friendly and spiritual consolation.

Dr. Dan Grigore shared with me his knowledge of supersymmetry and provided many useful references. The same goes to my friend Markus Gut. I have also had the privilege to enjoy exciting conversations with Stéphane Félix in Zürich .

Of course, I am grateful to my parents for their patience and *love.* Without them this work would never have come into existence (literally).

Finally, I wish to thank the following: Mario Germano (for his friendship and hospitality); Diane-Elise and Pascal Bettendorff (for their unmatched generosity); Amélie and Jonathan Chapuis, Rick, Maja, Paul, Phil, Michelle, Peter, Wendy, Scott, Eve, George, Andrew, Rob ... (for all the good and bad times we had together); Stephen and Bill (for special effects); Ludwig van Beethoven and Frank Sinatra (and they know why); *and* my brother (because he asked me to).

Lausanne, Switzerland — Simon Bossoney
December 7, 2004

Chapter 1

Fields satisfying the Wightman Axioms

1.1 Introduction

In this chapter we explain very generally what we mean by a "quantum field theory". Sometimes quantum fields are defined as quantum systems with infinitely many degrees of freedom. This makes the theory mathematically much more involved than in the case of quantum mechanics where one usually speaks of a theory with finitely many degrees of freedom.

More precisely, a quantum field should be able to describe a quantum "observable" depending on space-time points. Therefore one would wish to have self-adjoint operators acting on some Hilbert space and depending on space-time coordinates. Taking operator-valued functions on space-time would be too restrictive. Recall that even in the classical theory of electromagnetic fields one encounters field strengths which behave like $1/r^2$, whereas space-time contains the point $r = 0$. So these field strengths should strictly speaking be viewed as distributions rather than functions. Another argument in favor of distributions is that the classical field lagrangian is a functional

having the fields as well as their first time derivatives as arguments. Going over to the standard hamiltonian formalism one then obtains typical "Poisson-brackets" relations such as $\{\varphi(t,\vec{x}),\pi(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y})$, which in the "quantization" procedure should be replaced by equal-time commutation relations such as $\left[\varphi(t,\vec{x}),\pi(t,\vec{y})\right] = i\delta^3(\vec{x}-\vec{y})$. This would not be possible if both $\varphi(t,\vec{x})$ and $\pi(t,\vec{y})$ were functions. These arguments are motivations to define quantum fields as being operator-valued distributions rather than functions.
These ideas are precisely the content of the Wightman axioms.

1.2 The Field Axioms

Axiom 1. *There exists a separable complex Hilbert space of states $\mathcal{H}$ on which a positive definite scalar product $(.,.)$ is defined.*

As in usual quantum mechanics one can describe the state of the system, which would be a field, by a unit ray in the Hilbert space. Properties of the system are described by closed subspaces, observables by self-adjoint operators, etc...

Axiom 2. *There is a dense subspace $\mathcal{D}$ of $\mathcal{H}$ and a linear functional φ such that:*
1) $\varphi : \mathcal{S}(\mathbb{R}^4) \longrightarrow Lin(\mathcal{D},\mathcal{D})$
$f \longmapsto \varphi(f)$*, where* $\mathcal{S}(\mathbb{R}^4)$ *is the space of Schwartz functions,*
2) $\forall\Psi\ ,\chi \in \mathcal{D}$*,* $\forall f \in \mathcal{S}(\mathbb{R}^4)$
$f \longmapsto (\Psi,\varphi(f)\chi)$ *is a tempered distribution,*
3) $\forall\Psi,\Phi \in \mathcal{D}, \forall f \in \mathcal{S}(\mathbb{R}^4)$ *we have* $(\Psi,\varphi(f)\Phi) = (\varphi(\bar{f})\Psi,\Phi)$*.*

Some comments are in order. The so-defined object $\varphi(.)$ is what one calls the field. Usually one writes $\varphi(x)$, where "x" has the same significance as for a conventional distribution. More precisely $(\Psi, \varphi(x)\chi)$ is a distribution $\forall\Psi,\ \chi \in \mathcal{D}$. This justifies also the notation $\varphi(f) = \int dx f(x)\varphi(x)$. But one has to recall that since the operator $\varphi(f)$ is not required to be bounded, $\varphi(x)$ is in general not an "operator-valued" distribution unless it is "sandwiched" by two vectors in $\mathcal{D}$. But for a fixed Ψ in $\mathcal{D}$, one can prove that $\varphi(x)\Psi$ is a vector-valued distribution.

The third condition states that if f is a real Schwartz function then $\varphi(f)$ is a symmetric operator. It is in general a technically involved task to see if there is at least one self-adjoint extension of $\varphi(f)$ for all real $f \in \mathcal{S}(x)$ so we will generally admit that this is the case.

For a quantum field theory, we would like to have more. The description of the physical system should be Lorentz-, or Poincaré-invariant, at least quantitatively. Therefore, if there are two coordinate systems related to each other by a Poincaré transformation the states of the system relative to each observer should be linked by an operator in the Hilbert space which preserves the probability amplitudes for a given test. Otherwise stated, one would like to have a continuous unitary or anti-unitary representation of the Poincaré group. Since one usually works with the proper orthochronous component of the Poincaré group $L_+^\uparrow \rtimes \mathbb{R}^4$, the representation is unitary and we have the following

Axiom 3. *There is a continuous unitary representation* $U(\Lambda, a)\ ;(\Lambda, a) \in L_+^\uparrow \rtimes \mathbb{R}^4$ *of the proper orthochronous Poincaré group in* $\mathcal{H}$ *such that* $U(\Lambda, a) = U(\mathbb{1}, a)U(\Lambda, 0)$. *This representation may be two-valued on some subspace of* $\mathcal{H}$.
The field operators satisfy: $U^{-1}(\Lambda, a)\varphi((\Lambda, a)x)U(\Lambda, a) = \varphi(x)$

The last equation follows from the following reasoning. Suppose $\Phi,\ \Psi \in \mathcal{H}$ are such that $\phi = U(\Lambda, a)\Psi$ for some $(\Lambda, a) \in L_+^\uparrow \rtimes \mathbb{R}^4$. Let us write $y = (\Lambda, a)x$. Then, since $\varphi(x)$ is a scalar field, the mean value, which is an observable, $(\Phi, \varphi(y)\Phi)$ should be equal to $(\Psi, \varphi(x)\Psi)$, from which one draws the conclusion. This is simply the relativistic invariance of observable quantities.

Now, since that we want a relativistic physical theory, one should also want something like causality. In classical relativistic theory, two events x and y which are space-like separated must not "influence" each other. In quantum physics one speaks rather about states and properties of a system than about events. But, in a quantum field theory, the object $\varphi(x)$ defines an observable, and therefore a property, as soon as it is smeared out with a test-function f. The property is "located" on the support of f. Suppose one is given two functions f_1 and f_2 such that every point of the support of f_1 is space-like to every point of the support of f_2. We note this $supp f_1 \sim supp f_2$. Then the two observables $\varphi(f_1)$ and $\varphi(f_2)$ should be "independent". This means that the measurement of $\varphi(f_2)$ should not change the value of $\varphi(f_1)$ if the system is in a state in which this value is "actual". More precisely, if the system is in an Eigen-state $\Psi(1)$ of $\varphi(f_1)$ and the measurement of $\varphi(f_2)$ results in an Eigen-state $\Psi(2)$ of $\varphi(f_2)$, then this state must still be an Eigen-state for $\varphi(f_1)$ with the same Eigen-value as $\Psi(1)$. Otherwise said the observables $\varphi(f_1)$ and $\varphi(f_2)$ must be compatible:

Axiom 4. *If* $x \sim y$ *then* $[\varphi(x), \varphi(y)] = 0$.

This axiom is also called micro-causality. Of course it does not forbid EPR-like correlations.

Now we must say something about the Hilbert space we are working with. One would like to have a particular state: the ground state. This state, also called the vacuum, should look the same to any inertial observer. From this state, one should also be able to "build" any other state of the system. We precise this by the following

Axiom 5. *In $\mathcal{H}$ there is a particular state Ω, unique up to multiplications by a complex number, such that:*

1) $U(\Lambda, a)\Omega = \Omega$, $\forall(\Lambda, a) \in L_+^\uparrow \rtimes \mathbb{R}^4$,

2) $(\Omega, \Omega) = 1$,

3) Ω *is a cyclic vector in* $\mathcal{H}$,

4) $\lim_{\vec{a}\to 0} \lambda^3 \int d^3a U(\mathbb{1}, \vec{a})\chi(\lambda\vec{a})\Psi = (\Omega, \Psi) \int d^3a \chi(\vec{a})\Omega$,

$\forall\chi(\vec{a}) \in \mathcal{S}(x^3)$ *and* $\forall\Psi \in \mathcal{H}$.

In 3) "cyclic" means that vectors of the form $\varphi(f_1)\ldots\varphi(f_k)\Omega$ with $f_i \in \mathcal{S}\forall i = 1, \ldots k$ are dense in $\mathcal{H}$.

Condition 4) is called the "weak clustering" condition.

Some restrictions should also be imposed on the spectrum of the energy-momentum for the states. Axiom 3) gives the theory a unitary representation of the Poincaré group. In particular, to any translation four-vector a corresponds a unitary operator $U(\mathbb{1}, a)$. Since this sub-group of the Poincaré group is a four-dimensional commuting Lie-group we know that there exist four commuting self-adjoint operators, P^μ, such that $U(\mathbb{1}, a) = exp(-ia_\mu P^\mu)$. In classical field theory these generators of the translation group are known to represent the energy-momentum of the system, and we want this to remain true in the quantum theory. They are required to match the

Axiom 6. *There is a spectral decomposition* $P^\mu = \int p^\mu dE$ *such that:*

1) $p^\mu \in V_+$, *with* $V_+ \equiv \{k^\mu | k^\mu k_\mu \geq 0, k^0 \geq 0\}$,

2) $p^\mu = 0$ *is an isolated Eigen-value whose corresponding Eigen-space is one-dimensional and corresponds to* Ω.

Observe that since $p^\mu = 0$ is an isolated Eigen-value and since the spectrum of P^μ has to be Lorentz invariant there must exist a real number m^2 such that the spectrum must be of the form $\{p^\mu = 0\} \cup \bar{V}_+^m$, where $\bar{V}_+^m \equiv \{k^\mu | k^\mu \in V_+, k^\mu k_\mu \geq m^2\}$. We say then that there is a "mass-gap" in the spectrum.

Remark 1.2.1. Condition 4 in axiom 5 is redundant with this axiom. Indeed one can show [17] that, in the presence of a mass-gap, axiom 6 implies condition 4 of axiom 5. In the case of non-massive fields, i.e. fields with no mass-gap, one has to keep this condition, but then the spectrum of P^μ is only restricted to be in V_+, and $p^\mu = 0$ is not an isolated Eigen-value anymore.

These are the properties one reasonably can expect for a scalar hermitian quantum field. If one is working with a scalar field $\varphi(x)$ which is not hermitian, then one can define the fields $\varphi_1(x) \equiv 1/2(\varphi(x) + \varphi^\dagger(x))$ and $\varphi_2(x) \equiv i/2(\varphi(x) - \varphi^\dagger(x))$ which are both hermitian and from which one may recover the original field.
There are other fields of importance in a physical quantum field theory such as spinor fields $\psi_a(x)$, vector fields $A^\mu(x)$, or mixed fields $F_a^\mu(x)$. For the first, little has to be modified whereas for the other two there are some subtleties to be taken care of. We will treat both cases in the free case with more details.

1.3 The Free Fields

We will now explicitly construct various quantum fields in the free case. Free fields are always explicit and rigorous examples of fields which satisfy the Wightman axioms. Unfortunately, as soon as the dimension of space-time exceeds 3, they are the only known ones. Besides this, free fields are very important for the computations of the scattering processes, as will be explained in sections 2 and 3.

1.3.1 The Klein-Gordon Field

First of all we must consider a Hilbert space. Let $\mathcal{H}_1$ be the Hilbert space $L^2(\mathbb{R}^3; \frac{d^3p}{2\omega(p)})$, where $\omega(p) \equiv \sqrt{m^2 + \vec{p}^2}$. Let $\mathcal{H}_2 \equiv \mathcal{H}_1 \otimes \mathcal{H}_1$ and $\mathcal{H}_0 \equiv \mathbb{C}$. Similarly let $\mathcal{H}_n \equiv \mathcal{H}_1^{\otimes n}$. On each $\mathcal{H}_n$ one defines the symmetrization operator S_n^+ as follows:

$$S_n^+ \Phi_n(p_1, \ldots, p_n) = \frac{1}{n!} \sum_{\pi \in \sigma(n)} \Phi_n(p_{\pi(1)}, \ldots, p_{\pi(n)}), \forall \Phi_n \in \mathcal{H}_n. \tag{1.3.1}$$

Note that S_n^+ is a self-adjoint projection operator on $\mathcal{H}_n$. We can therefore define the spaces $\mathcal{H}_n^+ \equiv S_n^+ \mathcal{H}_n$. Our Hilbert-space is defined by:

$$\mathcal{H} \equiv \oplus_{n=0}^{\infty} \mathcal{H}_n^+. \tag{1.3.2}$$

An element Ψ in $\mathcal{H}$ can then be written as $\Psi = (\Psi_0, \Psi_1, \ldots)$ and the scalar product is $(\Psi, \Phi) = \sum_{n=0}^{\infty} (\Psi_n, \Phi_n)$. Of course all elements Φ should satisfy $\|\Phi\|^2 = \sum_{n=0}^{\infty} (\Phi_n, \Phi_n) < \infty$. The first axiom is then satisfied.

Now we have to construct the fields. First, one has to define the dense subspace $\mathcal{D}$. We define it as to be the set of all sequences $(\Phi_0, \Phi_1(p_1), \ldots)$ in $\mathcal{H}$, where all but a finite number of the $\Phi_n \in \mathcal{S}(\mathbb{R}^{3n})$ vanish. On this subspace one defines the following creation and annihilation operators:

$$\begin{aligned} a^\dagger(\vec{q}) : \mathcal{D}_n &\longrightarrow \mathcal{D}_{n+1}^\sharp, \\ (a^\dagger(\vec{q})\Phi_n)(\vec{k}_1, \ldots, \vec{k}_{n+1}) &= \frac{2\omega(q)}{\sqrt{n+1}} \sum_{j=1}^{n+1} \delta^3(\vec{q} - \vec{k}_j) \Phi_n(\vec{k}_1, \ldots, \widehat{\vec{k}_j}, \ldots, \vec{k}_{n+1}); \\ a(\vec{q}) : \mathcal{D}_n &\longrightarrow \mathcal{D}_{n-1}; n, \geq 1 \\ (a(\vec{q})\Phi_n)(\vec{k}_1, \ldots, \vec{k}_{n-1}) &= \sqrt{n} \Phi_n(\vec{q}, \vec{k}_1, \ldots, \vec{k}_{n-1}), \\ a(\vec{q})\Phi_0 &= 0. \end{aligned} \tag{1.3.3}$$

These operators satisfy the commutation relation:

$$\begin{aligned}[a(\vec{q}), a(\vec{k})] =&[a^\dagger(\vec{q}), a^\dagger(\vec{k})] = 0,\\ [a(\vec{q}), a^\dagger(\vec{k})] =&2\omega(q)\delta^3(\vec{q}-\vec{k}).\end{aligned} \tag{1.3.4}$$

Let $f \in \mathcal{D}_1$. Then one defines the "smeared-out" operators

$$\begin{aligned}a^\sharp(f) \equiv& \int \frac{d^3q}{2\omega(q)} a^\sharp(\vec{q}) f(\vec{q}),\\ a^\sharp(\vec{q}) =& a^\dagger(\vec{q}) \text{ or } a(\vec{q}).\end{aligned} \tag{1.3.5}$$

If we note $a^\sharp(f)|_n$ the restriction of $a^\sharp(f)$ to the subspace $\mathcal{D}_n$ we have

$$\|a(f)|_n\| = \|a^\dagger(f)|_{n-1}\| = \sqrt{n}\|f\|_{\mathcal{H}_1} \tag{1.3.6}$$

and the $a^\sharp(f)|_n\sqrt{n+1}$ are therefore bounded operators. Furthermore if $\Phi \in \mathcal{D}$ and $z \in \mathbb{C}$ then $\sum_{n=0}^{\infty} \frac{|z|^n}{n!}\|a^\sharp(f)^n\Phi\| < \infty$. Following Nelson [18] this means that $\mathcal{D}$ is a dense set of analytic vectors for $a(\bar{f})$, $a^\dagger(f)$ and $a(\bar{f}) + a^\dagger(f)$. Since $a^\dagger(f) \subset a(\bar{f})^\dagger$ the operators $a(\bar{f}) + a^\dagger(f)$ are essentially self-adjoint on $\mathcal{D}$. The field $\varphi(x)$ is defined as

$$\varphi(x) \equiv \frac{1}{(2\pi)^2}\int \frac{d^3k}{2\omega(k)}(a(\vec{k})e^{-i\langle(\omega(k),\vec{k}),x\rangle} + a^\dagger(\vec{k})e^{i\langle(\omega(k),\vec{k}),x\rangle}), \tag{1.3.7}$$

where $\langle p, x\rangle \equiv p^\mu\eta_{\mu\nu}x^\nu$ is the Lorentz-invariant scalar product. We take $\eta_{\mu\nu} = diag(1,-1,-1,-1)$. Now let $f \in \mathcal{S}(\mathbb{R}^4)$ be a real function. We get:

$$\begin{aligned}\varphi(f) =& \int d^4x f(x)\varphi(x),\\ =&\frac{1}{(2\pi)^4}\int d^4x \int d^4q \tilde{f}(q)e^{-i\langle q,x\rangle}\int \frac{d^3k}{2\omega(k)}(a(\vec{k})e^{-i\langle(\omega(k),\vec{k}),x\rangle} + a^\dagger(\vec{k})e^{i\langle(\omega(k),\vec{k}),x\rangle}),\\ =&\int d^4q \int \frac{d^3k}{2\omega(k)}(\delta(q+k)\tilde{f}(q)a(\vec{k}) + \delta(q-k)\tilde{f}(q)a^\dagger(\vec{k})),\\ =&\int \frac{d^3k}{2\omega(k)}(\tilde{f}(-\omega(k),-\vec{k})a(\vec{k}) + \tilde{f}(\omega(k),\vec{k})a^\dagger(\vec{k})),\\ =&a(\bar{\tilde{f}}) + a^\dagger(\tilde{f}),\end{aligned} \tag{1.3.8}$$

which is an essentially self-adjoint operator on $\mathcal{D}$. The other conditions of axiom 2 are easily seen to be satisfied by the $\varphi(x)$ defined in this way. Note that $\varphi(x)$ is, in the sense of distributions, a solution of the Klein-Gordon equation $(\Box + m^2)\varphi(x) = 0$. This is, in a nutshell, the reason one is talking about the free scalar field.

Now we have to find a continuous unitary representation of the proper orthochronous Poincaré group. We first define it on the subspace $\mathcal{D}_1$ which is dense in $\mathcal{H}_1$. Then this action, as it is unitary, can be uniquely extended to the whole Hilbert-space $\mathcal{H}_1$ and to $\mathcal{H}$ by taking tensor products. To make the Poincaré invariance apparent we are going to define it as follows: First note that a function $f \in \mathcal{D}_1$ can be extended to a function $\hat{f} \in \mathcal{S}(\mathbb{R}^4)$ by declaring it to be equal to f on the positive mass shell:

$$\hat{f}(\omega(p), \vec{p}) = f(\vec{p}). \tag{1.3.9}$$

This extension is not unique and therefore one should build equivalence classes by declaring $\hat{f} \sim \hat{g}$ if $\hat{f}$ and $\hat{g}$ are both extensions of the same f. Then we have a one-to-one relation between these equivalence classes and $\mathcal{D}_1$. The scalar product defined on $\mathcal{H}_1$ is then carried over in the following rather obvious way:

$$(\hat{f}(p), \hat{g}(p)) \equiv \int d^4p\delta^4(p^2 - m^2)\Theta(p_0)\overline{\hat{f}(p)}\hat{g}(p). \tag{1.3.10}$$

Here multiplication of $\delta^4(p^2 - m^2)$ by $\Theta(p_0)$ causes no troubles because the support of $\delta^4(p^2 - m^2)$ does not interfere with the support of the singular behavior of $\Theta(p_0)$. It can be read off the definition that this scalar product does not depend on the representative of the equivalence classes of $\hat{f}(p)$ nor on that of $\hat{g}(p)$. Furthermore, it is obvious that $(\hat{f}(p), \hat{g}(p)) = (f(\vec{p}), g(\vec{p}))$, but the advantage of using the first notation is its manifest covariance under a proper Poincaré transformation:

$$
\begin{aligned}
(\hat{f}(p), \hat{g}(p)) &= \int d^4p\delta^4(p^2 - m^2)\Theta(p_0)\overline{\hat{f}(p)}\hat{g}(p) \\
&= \int d^4p\delta^4(p^2 - m^2)\Theta(p_0)\overline{e^{i\langle p,a\rangle}\hat{f}(\Lambda^{-1}p)}e^{i\langle p,a\rangle}\hat{g}(\Lambda^{-1}p) \\
&\equiv \big((U(\Lambda, a)\hat{f})(p), (U(\Lambda, a)\hat{g})(p)\big).
\end{aligned}
\tag{1.3.11}
$$

Therefore the so-defined representation $U(\Lambda, a)$ is unitary and transposes by 1.3.9 to a unitary representation on $\mathcal{H}_1$. This representation carries over to $\mathcal{H}_n^+$ by the tensor product of operators and acts like $\mathbb{1}$ on $\mathcal{H}_0$.

The annihilation operator transforms as:

$$
\begin{aligned}
\big(U(\Lambda, a)a(\vec{p})\Phi_n\big)(\vec{k}_1, \ldots, \vec{k}_{n-1}) =& \sqrt{n}e^{i\langle \sum_{i=1}^{n-1} k_i, a\rangle}\Phi_n(\vec{p}, \Lambda^{-1}k_1, \ldots, \Lambda^{-1}\vec{k}_{n-1})|_{k_i^0=\omega(k_i)} \\
=& \sqrt{n}e^{-i\langle \Lambda p|_{p^0=\omega(p)}, a\rangle}\big(U(\Lambda, a)\Phi_n\big)(\vec{\Lambda p}|_{p^0=\omega(p)}, \vec{k}_1, \ldots, \vec{k}_{n-1}) \\
=& e^{-i\langle \Lambda p|_{p^0=\omega(p)}, a\rangle}\big(a(\vec{\Lambda p}|_{p^0=\omega(p)})U(\Lambda, a)\Phi_n\big)(\vec{k}_1, \ldots, \vec{k}_{n-1}),
\end{aligned}
\tag{1.3.12}
$$

from which one concludes:

$$
U(\Lambda, a)a(\vec{p})U^{-1}(\Lambda, a) = e^{-i\langle \Lambda p|_{p^0=\omega(p)}, a\rangle}a(\vec{\Lambda p}|_{p^0=\omega(p)}), \tag{1.3.13}
$$

and similarly for the creation operator

$$
U(\Lambda, a)a^\dagger(\vec{p})U^{-1}(\Lambda, a) = e^{i\langle \Lambda p|_{p^0=\omega(p)}, a\rangle}a^\dagger(\vec{\Lambda p}|_{p^0=\omega(p)}). \tag{1.3.14}
$$

With these two transformation properties one can verify that the field $\varphi(x)$ has the desired transformation properties.

Let us now check the causal commutation property of the free Klein-Gordan scalar field. The commutation relations of the creation and annihilation operators yield:

$$
\begin{aligned}
[\varphi(x),\varphi(y)] =& \frac{1}{(2\pi)^4}\int \frac{d^3p}{2\omega(p)}\frac{d^3q}{2\omega(q)}\big(e^{-(i\langle x,p\rangle)+(i\langle y,q\rangle)}[a(\vec{p}),a^\dagger(\vec{q})] \\
&+e^{(i\langle x,p\rangle)-(i\langle y,q\rangle)}[a^\dagger(\vec{p}),a(\vec{q})]\big) \\
=&\frac{1}{(2\pi)^4}\int \frac{d^3p}{2\omega(p)}\frac{d^3q}{2\omega(q)}2\omega(q)\delta^3(\vec{p}-\vec{q})\big(e^{-(i\langle x,p\rangle)+(i\langle y,q\rangle)} \\
&-e^{(i\langle x,p\rangle)-(i\langle y,q\rangle)}\big) \\
=&\frac{1}{(2\pi)^4}\int \frac{d^3p}{2\omega(p)}\big(e^{-(i\langle x-y,p\rangle)}-e^{(i\langle x-y,p\rangle)}\big) \\
=&\frac{1}{(2\pi)^4}\int \frac{d^3p}{\omega(p)}-i\sin(\langle x-y,p\rangle) \\
\equiv& -iD_m(x-y).
\end{aligned}
\tag{1.3.15}
$$

To see that $D_m(x-y)$ vanishes for x and y spacelike we employ a Lorentz transformation which sends $x-y$ to $(0,\vec{d})$. $D_m(0,\vec{d})$ is then equal to the integration of an odd function over an even domain (recall that d^3p is the Lebesgue-measure, which is invariant under the change $\vec{p}\to-\vec{p}$), which gives 0, showing the causal property of the field commutation relations.

To meet axiom 5 one takes for the vacuum state Ω the number $1\in\mathcal{H}_0$. The first three statements of axiom 5 are immediately seen to hold. By 1.2.1 we only need to satisfy axiom 6 which is done by defining P^μ with the following rather natural equations:

$$
\begin{aligned}
(P^0\Phi_n)(\vec{k}_1,\dots,\vec{k}_n) \equiv& \sum_{j=1}^n \omega(k_j)\Phi_n(\vec{k}_1,\dots,\vec{k}_n), \\
(\vec{P}\Phi_n)(\vec{k}_1,\dots,\vec{k}_n) \equiv& \sum_{j=1}^n \vec{k}_j\Phi_n(\vec{k}_1,\dots,\vec{k}_n), \forall\phi_n\in\mathcal{H}_n^+; \\
P^0\Omega \equiv& 0\equiv\vec{P}\Omega.
\end{aligned}
\tag{1.3.16}
$$

From these equalities it follows that the P^μ are four commuting unbounded self-adjoint operators. Since they commute amongst themselves there exists a spectral measure $E(p)$ such that $P^\mu=\int p^\mu dE$ and, by the definition of $\omega(p)$, the mass-gap surely exists.

This establishes the existence of a free, scalar, and hermitian quantum field.

1.3.2 The Complex Scalar Field

Little has to be changed to build a complex scalar field. All axioms remain true except part 3) of axiom 2 which now reads $(\Phi\varphi^\dagger(\bar{f}), \Psi) = (\Phi, \varphi(f)\Psi)$: the field is no longer hermitian. However, one can build two hermitian fields, $f_1(x) \equiv \frac{1}{2}\big(\varphi(x)+\varphi^\dagger(x)\big)$ and $f_2(x) \equiv \frac{-i}{2}\big(\varphi(x)-\varphi^\dagger(x)\big)$, from which one can recover the field $\varphi(x) = f_1(x)+if_2(x)$. Therefore, taking the two fields $f_i(x)$ to be two free scalar hermitian fields which commute, one can write:

$$\begin{aligned}\varphi(x) =& f_1(x) + if_2(x)\\ =&\frac{1}{(2\pi)^2}\int\frac{d^3k}{2\omega(k)}\big([a_1(\vec{k}) + ia_2(\vec{k})]e^{(-i\langle(\omega(k),\vec{k}),x\rangle)}\\ &+[a_1^\dagger(\vec{k}) + ia_2^\dagger(\vec{k})]e^{(i\langle(\omega(k),\vec{k}),x\rangle)}\big)\\ \equiv&\frac{1}{(2\pi)^2}\int\frac{d^3k}{2\omega(k)}\big(a(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + b^\dagger(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)}\big).\end{aligned} \tag{1.3.17}$$

The Hilbert space is now equal to the tensor product of two copies of $\mathcal{H}$, and the operators $a^\sharp(\vec{p})$ and $b^\sharp(\vec{q})$ act on it as the creation and annihilation operators of two different types of particles (actually a particle and its anti-particle). The commutation relations are:

$$\begin{aligned}[\varphi(x), \varphi(y)] =&[\varphi^\dagger(x), \varphi^\dagger(y)] = 0\\ [\varphi(x), \varphi^\dagger(y)] =& -2iD_m(x-y).\end{aligned} \tag{1.3.18}$$

One can easily check that all the Wightman axioms are fulfilled.

1.3.3 The Scalar Ghost Field

In the previous two paragraphs, we met scalar fields satisfying causal commutation relations. This is forced upon us by the well-known "spin and statistics-theorem". In the case of scalar fields this theorem precisely states:

Theorem 1.3.1. *If a scalar field $\varphi(x)$ satisfies causal anti-commutation relations, i.e.*

$$\begin{aligned} \{\varphi(x), \varphi(y)\} =& 0, \\ \{\varphi(x), \varphi^\dagger(y)\} =& 0, \ \textit{if } x \sim y, \end{aligned} \tag{1.3.19}$$

then $\varphi(x) = 0$.

Proof. A proof can be found in [27] □

But we can (and as we will see must) nevertheless construct "anti-commuting" scalar fields, by abandoning axiom 4. As this axiom was introduced for physical reasons, these anti-commuting scalar fields are considered to be unphysical-hence the name "ghost-fields". First, we are going to introduce a new Hilbert-space. The "vacuum-sector" $\mathcal{H}_0$ will again be $\mathbb{C}$. We then define $\mathcal{H}_n^- \equiv S^-\mathcal{H}^{\otimes n}$ where S^- is the anti-symmetrization operator defined by

$$(S^-\Phi_n)(\vec{k}_1, \ldots, \vec{k}_n) = \frac{1}{n!} \sum_{\pi\in\sigma(n)} (-1)^{i+1} \Phi_n(\vec{k}_{\pi(1)}, \ldots, \vec{k}_{\pi(n)}). \tag{1.3.20}$$

The Hilbert-space is then $\mathcal{H} \equiv \bigoplus_{n,m=0}^{\infty} \mathcal{H}_n^- \otimes \mathcal{H}_m^-$. On this Hilbert-space we define two creation and annihilation operators:

$$
\begin{aligned}
\big(a(\vec{p})\Phi_n \otimes \Psi_m\big)(\vec{k}_1,\ldots,\vec{k}_{n-1},\vec{l}_1,\ldots,\vec{l}_m) =& \sqrt{n}\Phi_n \otimes \Psi_m(\vec{p},\vec{k}_1,\ldots,\vec{k}_{n-1},\vec{l}_1,\ldots,\vec{l}_m),\\
\big(a^\dagger(\vec{p})\Phi_n \otimes \Psi_m\big)(\vec{k}_1,\ldots,\vec{k}_{n+1},\vec{l}_1,\ldots,\vec{l}_m) =& \frac{2\omega(p)}{\sqrt{n+1}}\sum_{i=1}^{n+1}(-1)^{i+1}\delta^3(\vec{p}-\vec{k}_i)\\
&\times\Phi_n \otimes \Psi_m(\vec{k}_1,\ldots,\widehat{\vec{k}_i},\ldots,\vec{k}_{n+1},\vec{l}_1,\ldots,\vec{l}_m);\\
\big(b(\vec{p})\Phi_n \otimes \Psi_m\big)(\vec{k}_1,\ldots,\vec{k}_n,\vec{l}_1,\ldots,\vec{l}_{m-1}) =& (-1)^n\sqrt{m}\Phi_n \otimes \Psi_m(\vec{k}_1,\ldots,\vec{k}_n,\vec{p},\vec{l}_1,\ldots,\vec{l}_{m-1}),\\
\big(b^\dagger(\vec{p})\Phi_n \otimes \Psi_m\big)(\vec{k}_1,\ldots,\vec{k}_n,\vec{l}_1,\ldots,\vec{l}_{m+1}) =& \frac{2\omega(p)}{\sqrt{m+1}}\sum_{i=1}^{m+1}(-1)^{i+1+n}\delta^3(\vec{p}-\vec{l}_i)\\
&\times\Phi_n \otimes \Psi_m(\vec{k}_1,\ldots,\vec{k}_n,\vec{l}_1,\ldots,\widehat{\vec{l}_i},\ldots,\vec{l}_{m+1}).
\end{aligned}
\tag{1.3.21}
$$

Placing a $(-1)^n$ in front of the second couple of operators is called a "Krein-transformation", and ensures the following anti-commutation relations:

$$
\begin{aligned}
\{a^\sharp(\vec{p}),&b^\sharp(\vec{q})\} = 0,\\
\{a(\vec{p}),a(\vec{q})\} =& \{b(\vec{p}),b(\vec{q})\} = 0,\\
\{a^\dagger(\vec{p}),a^\dagger(\vec{q})\} =& \{b^\dagger(\vec{p}),b^\dagger(\vec{q})\} = 0,\\
\{a^\dagger(\vec{p}),a(\vec{q})\} =& \{b^\dagger(\vec{p}),b(\vec{q})\} = 2\omega(p)\delta^3(\vec{p}-\vec{q}).
\end{aligned}
\tag{1.3.22}
$$

We can now define the following ghost-fields:

$$
\begin{aligned}
u(x) =& \frac{1}{(2\pi)^2}\int\frac{d^3k}{2\omega(k)}(b(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + a^\dagger(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)}),\\
\tilde{u}(x) =& \frac{1}{(2\pi)^2}\int\frac{d^3k}{2\omega(k)}(-a(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + b^\dagger(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)}).
\end{aligned}
\tag{1.3.23}
$$

Note that $\tilde{u}(x)$ is not the adjoint of $u(x)$, but its virtue lies in its causal anti-commutation relations with the latter, as can be easily computed from the definitions:

$$
\begin{aligned}
\{u(x),\tilde{u}(y)\} =& -iD_m(x-y),\\
\{u(x),u(y)\} =& \{\tilde{u}(x),\tilde{u}(y)\} = 0.
\end{aligned}
\tag{1.3.24}
$$

We also have

$$\{u(x), u^\dagger(y)\} = \frac{2}{(2\pi)^4}\int \frac{d^3k}{\omega(k)} cos(\langle p, x-y\rangle), \tag{1.3.25}$$

which is an accusal function, even though it is Poincaré-invariant. Accusal means it does not vanish if $x-y$ is space-like, as expected by the spin and statistics theorem.

Remark 1.3.1. The distribution $D_m(x-y)$ is called the Jordan-Pauli function. It is possible to reconstruct the whole field with this function and the "initial conditions" $u(x)$ and $\partial_0 u(x)$. Indeed, a fairly simple but rather lengthy computation shows that

$$u(x) = 2\pi \int_{y^0=x^0} d^3y D(x-y) \overleftrightarrow{\partial_0^y} u(y). \tag{1.3.26}$$

This equation is generally valid for any scalar quantum field.

1.3.4 Dirac and Majorana Fields

We arrive now at the very important spinor fields. Before we construct them properly, we have to do a minimal amount of representation theory of the Lorentz group.

As is surely known by the reader, the Lorentz group is a non-connected Lie group whose proper orthochronous component has six generators $M^{\mu\nu} = -M^{\nu\mu}$ satisfying the following commutation relations:

$$[M^{\mu\nu}, M^{\alpha\beta}] = -i(\eta^{\mu\alpha}M^{\nu\beta} + \eta^{\nu\beta}M^{\mu\alpha} - \eta^{\mu\beta}M^{\nu\alpha} - \eta^{\nu\alpha}M^{\mu\beta}). \tag{1.3.27}$$

A Lorentz transformation Λ can be written as $\Lambda = exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are real numbers corresponding to boosts, $\mu = 0,$ and $\nu = 1, 2, 3$, or rotations $\mu, \nu = 1, 2, 3$.

Our aim now is to build finite-dimensional, faithful, and irreducible representations of this Lie group. Let us consider the Pauli matrices σ^μ. If we make the correspondence

$$M^{0j} \mapsto \frac{-i}{2}\sigma^j; \quad M^{ij} \mapsto \frac{1}{2}\epsilon^{ijk}\sigma^k; \quad i, j, k = 1, 2, 3, \tag{1.3.28}$$

one can verify that the the commutation relations of the Lorentz generators remain valid. We therefore have a faithful representation

$$\Lambda = exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) \longmapsto exp(\frac{1}{2}\omega_{oj}\sigma^j + \frac{i}{4}\omega_{ij}\epsilon^{ijk}\sigma^k) \equiv A(\Lambda) \in SL(2,\mathbb{C}). \qquad (1.3.29)$$

Any matrix in $SL(2,\mathbb{C})$ can be written in this way, but this mapping is not $1:1$ but $1:2$. Indeed, for a rotation $O(1,\theta)$ along the $1-$axis, which is of course contained in the Lorentz group, one has $O(1,\theta) = e^{i\theta M^{23}} \to e^{\frac{i\theta}{2}\sigma^1}$, but the matrix $e^{i\frac{\theta+2\pi}{2}\sigma^1} \neq e^{\frac{i\theta}{2}\sigma^1}$ corresponds to the same rotation in the Lorentz group.

The representation 1.3.29 ,which is realized on the vector space $\mathbb{C}^2$, is called the spinor $D^{(\frac{1}{2},0)}$ representation. An element $u_a \in \mathbb{C}^2,\ a=1,2$ is called a spinor.

There is another representation which is inequivalent to the preceding one: the so-called $D^{(0,\frac{1}{2})}$ representation. It is defined by taking the complex conjugate of 1.3.29 :

$$\Lambda \longmapsto A^*(\Lambda) = e^{(\frac{1}{2}\omega_{oj}\sigma^{*j} - \frac{i}{4}\omega_{ij}\epsilon^{ijk}\sigma^{*k})}. \qquad (1.3.30)$$

This representation is also realized on $\mathbb{C}^2$, but a spinor transforming according to 1.3.30 is noted $\bar{u}_{\bar{a}},\ \bar{a}=1,2$.

On these two representation spaces one can define a nondegenerate $SL(2,\mathbb{C})$-invariant inner product. Consider the matrix

$$\epsilon^{ab} = \epsilon^{\bar{a}\bar{b}} = -\epsilon_{ab} = -\epsilon_{\bar{a}\bar{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We then define the inner product by $\langle u,v\rangle \equiv u_a v_b \epsilon^{ba} \equiv u^a v_a$ and $\langle \bar{u},\bar{v}\rangle \equiv \bar{u}^{\bar{a}}\bar{v}^{\bar{b}}\epsilon_{\bar{b}\bar{a}} \equiv u_{\bar{a}}v^{\bar{a}}$. This inner product is indeed invariant since $\langle u,v\rangle = det(u,v) \longmapsto \langle Au,Av\rangle = det(Au,Av) = det[A(u,v)] = detAdet(u,v) = \langle u,v\rangle$, where we used $A \in SL(2,\mathbb{C})$.

Remark 1.3.2. The signification of the positions of the indices on the ϵ-matrix is as for the η-matrix in the formalism of special relativity: a repeated index is summed and one raises and lowers the indices using the η-matrix.

Higher dimensional representations of the Lorentz group are obtained by constructing the tensor product. Consider the tensor product space

$$u_{a_1\ldots a_n} \in (\mathbb{C}^{\otimes n}) \tag{1.3.31}$$

carrying the representation

$$(D^{(\frac{1}{2},0)})^{\otimes n} : u_{a_1\ldots a_n} \longmapsto A_{a_1}^{b_1} \ldots A_{a_n}^{b_n} u_{b_1\ldots b_n}. \tag{1.3.32}$$

This representation is not irreducible. Indeed, taking a vector $u_{a_1\ldots a_n}$ antisymmetic in, say, the first two indcices is transformed by 1.3.32 into a vector $u_{b_1\ldots b_n}$ which is also antisymmetric in the first two indices:

$$\begin{aligned} u_{a_1\ldots a_n} =& A_{a_1}^{b_1} \ldots A_{a_n}^{b_n} u_{b_1\ldots b_n} \\ =& - A_{a_1}^{b_1} A_{a_2}^{b_2} \ldots A_{a_n}^{b_n} u_{b_2,b_1\ldots b_n} \\ =& - u_{a_2,a_1\ldots a_n}. \end{aligned} \tag{1.3.33}$$

Since the antisymmetric subspaces are easily seen to carry a representation which is not faithfull, only the symmetric subspaces can be irreducible faithfull representation of the Lorentz group, and indeed, they are [29].

The same holds for the complex conjugate tensor representations $(D^{(0,\frac{1}{2})})^{\otimes n}$. The spinors of rank (n, m) are defined to be the elements of

$$u_{a_1\ldots a_n,\bar{b}_1\ldots\bar{b}_m} \in D^{(\frac{n}{2},\frac{m}{2})} \equiv sym(\mathbb{C}^2)^{\otimes n} \otimes sym(\bar{\mathbb{C}}^2)^{\otimes m}, \tag{1.3.34}$$

symmetric in the indices $(a_1 \ldots a_n)$ and $(\bar{b}_1 \ldots \bar{b}_m)$ seperately. They carry the irreducible and faithfull representation

$$u_{b_1\ldots b_n,\bar{c}_1\ldots\bar{c}_m} \mapsto u_{a_1\ldots a_n,\bar{b}_1\ldots\bar{b}_m} = A_{a_1}^{b_1} \ldots A_{a_n}^{b_n} A_{\bar{b}_1}^{*\bar{c}_1} A_{\bar{b}_m}^{*\bar{c}_m} u_{b_1\ldots b_n,\bar{c}_1\ldots\bar{c}_m}. \tag{1.3.35}$$

In fact, all finite dimensional faithfull irreducible representations of the Lorentz group are isomorphic to one of the $D^{(\frac{n}{2},\frac{m}{2})}$ [29] , whose dimension are $(n+1)(m+1)$.

Remark 1.3.3. The defining representation of the Lorentz group is linked to the $D^{(\frac{1}{2},\frac{1}{2})}$ via the Pauli-matrices σ^μ $(=\eta^{\mu\nu}\sigma_\nu)$. A straightforward computation shows that

$$A(\Lambda)_a^b A(\Lambda)_{\bar{a}}^{*\bar{b}} \sigma^\mu_{b\bar{b}} = (\Lambda^{-1})^\mu_\nu \sigma^\nu_{a\bar{a}}. \tag{1.3.36}$$

Therefore one can make a correpondance between a four-vector x^μ and a $D^{(\frac{1}{2},\frac{1}{2})}$-spinor $\hat{x}_{a\bar{b}}$ through the equation $\hat{x}_{a\bar{b}} = x^\mu \sigma_{\mu a\bar{b}}$.

Going back to the $D^{(\frac{1}{2},0)}$ representation, one can write down an invariant (classical) field equation for a spinor field $\psi_a(x)$. Under a Poincaré-transformation such a field ransforms as:

$$\psi_a(x) \mapsto \psi'_b(y) = A(\Lambda)^a_b \psi_a(\Lambda^{-1}(y-a)). \tag{1.3.37}$$

Making use of equation 1.3.36 one can show that the quantities $\sigma^\mu \partial^x_\mu$ transforms as a bispinor of the tipe $(1,1)$. Therefore the differential equation

$$i\sigma^\mu_{a\bar{b}} \partial^x_\mu \psi^a(x) = 0 \tag{1.3.38}$$

is invariant. The left-hand side transforms as a spinor of the type $(0,1)$ and therefore, if one wants to have a non-vanishing righthandside involving only the spinor $\psi_a(x)$, one has no other choice, up to a constant, than to put

$$i\sigma^\mu_{a\bar{b}} \partial^x_\mu \psi^a(x) = -m\bar{\psi}_{\bar{b}}(x). \tag{1.3.39}$$

This is the Majorana equation, the field $\psi_a(x)$ beeing the Majorana field. Taking the complex conjugate of 1.3.39 one gets

$$i\sigma^\mu_{b\bar{a}} \partial^x_\mu \bar{\psi}^{\bar{a}}(x) = m\psi_b(x). \tag{1.3.40}$$

Remark 1.3.4. The indices of the Pauli-matrices $\sigma^{\mu}_{a\bar{b}}$ may also be raised using the ϵ-tensor: $\bar{\sigma}^{\mu\bar{c}d} \equiv \epsilon^{\bar{c}\bar{b}}\epsilon^{da}\sigma^{\mu}_{a\bar{b}}$. In this notation the Majorana equation reads:

$$i\bar{\sigma}^{\mu\bar{a}b}\partial^x_\mu\psi_b(x) = m\bar{\psi}^{\bar{a}}(x). \tag{1.3.41}$$

The factor m plays the role of the mass as can be read off the equation

$$\begin{aligned} -i\bar{\sigma}^{\nu\bar{b}c}\partial^x_\nu i\sigma^{\mu}_{a\bar{b}}\partial^x_\mu\psi^a(x) &\overset{1.3.39}{=} mi\bar{\sigma}^{\nu\bar{b}c}\partial^x_\nu\bar{\psi}_{\bar{b}}(x) \overset{1.3.41}{=} \\ -m^2\psi^c(x) &= \bar{\sigma}^{\nu\bar{b}c}\partial^x_\nu\sigma^{\mu}_{a\bar{b}}\partial^x_\mu\psi^a(x) = \\ \eta^{\mu\nu}\partial^x_\mu\partial^x_\nu\delta^c_a\psi^a(x) &= \Box\psi^c(x). \end{aligned} \tag{1.3.42}$$

Therefore one concludes that a Majorana field satisfies the Klein-Gordon equation with mass m, and is therefore a free field.
The defect of the Majorana equation is that it involves the spinor field $\psi_a(x)$ together with its complex conjugate $\bar{\psi}^{\bar{a}}$. One can remedy to this in the following way. One takes a spinor field $\varphi_a(x)$ which is only required to satisfy the Klein-Gordon equation. Then one takes a second spinor field $\chi^{\bar{b}}(x) \overset{\text{def}}{=} \frac{i}{m}\bar{\sigma}^{\mu\bar{b}a}\partial_\mu\varphi_a(x)$. We then get the equations:

$$\begin{aligned} i\bar{\sigma}^{\mu\bar{b}a}\partial_\mu\varphi_a(x) &= m\chi^{\bar{b}}(x), \\ i\sigma^{\mu}_{c\bar{b}}\partial_\mu\chi^{\bar{b}}(x) &= m\varphi_c(x). \end{aligned} \tag{1.3.43}$$

By defining the following objects

$$\psi(x) \overset{\text{def}}{=} \begin{pmatrix} \psi_a(x) \\ \chi^{\bar{b}}(x) \end{pmatrix}, \qquad \gamma^\mu \overset{\text{def}}{=} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{1.3.44}$$

this can be rewritten to give the well-known Dirac equation

$$i\gamma^\mu\partial_\mu\psi(x) = m\psi(x), \tag{1.3.45}$$

as well as the Dirac algegra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1.3.46}$$

Remark 1.3.5. A Dirac field can be seen as the sum of two Majorana fields. Indeed, if we define

$$\begin{aligned}\psi_a^{(1)}(x) &\stackrel{\text{def}}{=} \frac{1}{2}(\varphi_a(x) + \chi_a(x));\\ \psi_a^{(2)}(x) &\stackrel{\text{def}}{=} \frac{i}{2}(\varphi_a(x) - \chi_a(x)),\end{aligned} \tag{1.3.47}$$

one can verifie through a straightforward computation that both, $\psi^{(1)}$ and $\psi^{(2)}$, are Majorana fields. Furthermore

$$\psi(x) = \begin{pmatrix} \psi_a^{(1)}(x) - i\psi_a^{(2)}(x) \\ \bar{\psi}^{(1)\bar{b}}(x) - i\bar{\psi}^{(2)\bar{b}}(x). \end{pmatrix} \tag{1.3.48}$$

To build invariant bilinear products with Dirac spinors we will need the matrix

$$\beta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \delta_{\bar{b}}^{\bar{a}} \\ \delta_b^c & 0 \end{pmatrix}, \tag{1.3.49}$$

as well as the so-called Dirac-conjugate spinor

$$\bar{\psi}(x) \stackrel{\text{def}}{=} \psi^\dagger \beta = (\chi^b(x), \bar{\varphi}_{\bar{a}}(x)). \tag{1.3.50}$$

We can check the following algebraic rules:

$$\gamma^{\mu\dagger}\beta = \beta\gamma^\mu, \;\; \beta S^{-1}(\Lambda) \stackrel{\text{def}}{=} \beta \begin{pmatrix} A^{-1}(\Lambda) & 0 \\ 0 & A(\Lambda)^\dagger \end{pmatrix} = S^\dagger(\Lambda)\beta. \tag{1.3.51}$$

$S(\lambda)$ is important because a Dirac spinor $\psi(x)$ transforms as $\psi(x) \xrightarrow{\lambda} \psi'(y) = S(\lambda)\psi(\lambda^{-1}(y-a))$ under a Poincaré transformation (a, λ). We can then define the quantities

$$\begin{aligned}\bar{\psi}(x)\psi(x) =& \chi^b(x)\varphi_b(x) + \bar{\varphi}_{\bar{a}}(x)\bar{\chi}^{\bar{b}}(x),\\ \bar{\psi}(x)\gamma^\mu\psi(x) =& \chi^b(x)\sigma^\mu_{b\bar{a}}\bar{\chi}^{\bar{a}}(x) + \bar{\varphi}_{\bar{a}}(x)\bar{\sigma}^{\mu\bar{a}b}\varphi_b(x),\end{aligned} \tag{1.3.52}$$

which are easily seen to transform as a scalar, respectively a vector, under a Poincaré transformation.

The second quantity just defined has a vanishing divergence if $\psi(x)$ and $\varphi(x)$ are two solution to the Dirac equation. Indeed:

$$
\begin{aligned}
\partial_\mu \bar{\varphi}(x)\gamma^\mu \psi(x) =&(\partial_\mu \varphi^\dagger(x)\beta\gamma^\mu)\psi(x) + \bar{\varphi}(x)(\partial_\mu \gamma^\mu \psi(x)) \\
=&(\partial_\mu \varphi^\dagger(x)\gamma^{\mu\dagger}\beta)\psi(x) + \bar{\varphi}(x)(-im\psi(x)) \\
=&(\partial_\mu \gamma^\mu \varphi(x))^\dagger \beta\psi(x) - im\bar{\varphi}(x)\psi(x) \\
=&im\bar{\varphi}(x)\psi(x) - im\bar{\varphi}(x)\psi(x) = 0.
\end{aligned}
\tag{1.3.53}
$$

Therefore, if we note by Σ a space-like hyper-surface and by $d\sigma^\mu(x)$ its infinitesimal hyper-surface, the quantity $\int_\Sigma d\sigma_\mu(x)\bar{\varphi}(x)\gamma^\mu\psi(x)$ is an invariant scalar, due to the Gauss theorem, if both $\varphi(x)$ and $\psi(x)$ are Dirac-spinors. As a consequence one can define an invariant positive defined scalar product on the space of smooth Dirac-spinors:

$$
(\varphi(x), \psi(x)) \stackrel{\text{def}}{=} \int d^3x \varphi^\dagger(x)\psi(x) = \int_{\mathbb{R}^3} d^3x \bar{\varphi}(x)\gamma^0\psi(x). \tag{1.3.54}
$$

We therefore have a pre-Hilbert space which can in the usual way be completed in a Hilbert-space $\tilde{\mathcal{H}}_1$.

Let us now construct the general solution to the Dirac equation. Note first that if $\psi(x)$ is a solution to the Dirac equation, then

$$
\begin{aligned}
m^2\psi(x) =&i\gamma^\mu\partial_\mu i\gamma^\nu\partial_\nu\psi(x) = -g^{\mu\nu}\partial_\mu\partial_\nu\psi(x) \\
= -\Box\psi(x),&
\end{aligned}
\tag{1.3.55}
$$

so that $\psi(x)$ is also a solution to the Klein-Gordon equation with mass m. We may therefore write

$$\begin{aligned}\psi(x) =&\frac{1}{(2\pi)^2}\int d^4p\delta(p^2-m^2)\tilde{\psi}(p)e^{-ipx}\\=&\frac{1}{(2\pi)^2}\int\frac{d^3p}{2\omega_p}(\tilde{\psi}(\omega_p,\vec{p})e^{-ipx}+\tilde{\psi}(-\omega_p,\vec{p})e^{+i\omega_px^0-ip_ix^i})\\=&\frac{1}{(2\pi)^2}\int\frac{d^3p}{2\omega_p}(\tilde{\psi}(\omega_p,\vec{p})e^{-ipx}+\tilde{\psi}(-\omega_p,-\vec{p})e^{+ipx})\\\overset{\text{def}}{=}&\frac{1}{(2\pi)^2}\int\frac{d^3p}{2\omega_p}(\tilde{\psi}^+(p)e^{-ipx}+\tilde{\psi}^-(p)e^{+ipx}).\end{aligned}\tag{1.3.56}$$

The spinors $\tilde{\psi}^+$ and $\tilde{\psi}^-$ satisfy

$$(\gamma^\mu p_\mu - m)\tilde{\psi}^+(p) = 0,\ \ (\gamma^\mu p_\mu + m)\tilde{\psi}^-(p) = 0.\tag{1.3.57}$$

Obviously, if we look for plane-wave solutions, there are two linearly independent solutions for $(\gamma^\mu p_\mu + m)\tilde{\psi}^+(p) = 0$ and two others for $(\gamma^\mu p_\mu + m)\tilde{\psi}^-(p) = 0$. Let $m_\mu = (m, 0, 0, 0)$ and Λ_p be the pure boost such as $\Lambda^\mu_{p\nu} p_\mu = m_\nu$. Then for the positive frequency solutions $\psi^+_{sp}(x)$ we have:

$$\begin{aligned}\psi^+_{sp}(x) =&\chi(p)^+_s e^{-ipx} = S(\Lambda_p)\chi(m)^+_s e^{-im_\nu(\Lambda_p^{-1\nu}{}_\mu x^\mu)}\\=&S(\Lambda_p)\chi(m)^+_s e^{-ipx},\end{aligned}\tag{1.3.58}$$

where $\chi(m)^+_s$ is the solution to $(\gamma^\mu m_\mu - m)\chi(m)^+_s = 0$. They are very easy to find. Indeed:

$$\chi(m)^+_s = \sqrt{m}\begin{pmatrix}\chi_s\\\chi_s\end{pmatrix},\ \ \chi_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \ \chi_2 = \begin{pmatrix}0\\1\end{pmatrix}.\tag{1.3.59}$$

The so-defined plane-wave solutions are pseudo-orthonormal with respect to the previously defined scalar product:

$$\begin{aligned}
(\psi^+_{sp}(x), \psi^+_{tq}(x)) =& \int_{\mathbb{R}^3} d^3x \bar{\psi^+}_{sp}(x)\gamma^0\psi^+_{tq}(x) \\
=& \int_{\mathbb{R}^3} d^3x (\chi(m)^+_s)^\dagger S(\Lambda_p)^\dagger \beta\gamma^0 S(\Lambda_q)\chi(m)^+_t e^{-ix(q-p)} \\
=&(\chi(m)^+_s)^\dagger \beta S(\Lambda_p)^{-1}\gamma^0 S(\Lambda_p)\chi(m)^+_t \delta^3(\vec{p}-\vec{q}) \\
=&(\chi(m)^+_s)^\dagger \beta \Lambda^0_{p\mu}\gamma^\mu \chi(m)^+_t \delta^3(\vec{p}-\vec{q}) \\
=&\frac{1}{m}(\chi(m)^+_s)^\dagger \beta m_\nu \Lambda^\nu_{p\mu}\gamma^\mu \chi(m)^+_t \delta^3(\vec{p}-\vec{q}) \\
=&\frac{1}{m}(\chi(m)^+_s)^\dagger \begin{pmatrix} \bar{\sigma}^\mu p_\mu & 0 \\ 0 & \sigma^\mu p_\mu \end{pmatrix} \chi(m)^+_t \delta^3(\vec{p}-\vec{q}) \\
=&2\omega_p \delta_{st}\delta^3(\vec{p}-\vec{q}).
\end{aligned} \tag{1.3.60}$$

For the negative frequency solutions, note that

$$(\gamma^\mu m_\mu + m)\chi(m)^-_s = 0 \tag{1.3.61}$$

for

$$\chi(m)^-_s = \sqrt{m}\begin{pmatrix} \chi_s \\ -\chi_s \end{pmatrix}. \tag{1.3.62}$$

As before we take

$$\psi^-_{ps}(x) = S(\Lambda_p)\chi(m)^-_s e^{ipx}. \tag{1.3.63}$$

Again one has $(\psi^-_{ps}(x), \psi^-_{qt}(x)) = 2\omega_p \delta_{st}\delta^3(\vec{p}-\vec{q})$.

The scalar product between positive and negative frequency solutions reads:

$$\begin{aligned}(\psi_{sp}^-(x),\psi_{tq}^+(x)) =& \int_{\mathbb{R}^3} d^3x (\chi(m)_s^-)^\dagger S^\dagger(\Lambda_p)\beta\gamma^0 S(\Lambda_q)\chi(m)_t^+ e^{-ix(q+m)}\\ =&(\chi(m)_s^-)^\dagger S^\dagger(\Lambda_p)\beta\gamma^0 S(\Lambda_{\omega_p,-\vec{p}})\chi(m)_t^+\delta^3(\vec{q}+\vec{p})e^{-2ip_0x^0}\\ =&(\chi(m)_s^-)^\dagger S^\dagger(\Lambda_p) S(\Lambda_{\omega_p,-\vec{p}})\chi(m)_t^+\delta^3(\vec{q}+\vec{p})e^{-2ip_0x^0}\\ =&(\chi(m)_s^-)^\dagger S^\dagger(\Lambda_p) S^{-1}(\Lambda_p)\chi(m)_t^+\delta^3(\vec{q}+\vec{p})e^{-2ip_0x^0}\\ =&(\chi(m)_s^-)^\dagger S(\Lambda_p) S^{-1}(\Lambda_p)\chi(m)_t^+\delta^3(\vec{q}+\vec{p})e^{-2ip_0x^0}\\ =&0,\end{aligned} \tag{1.3.64}$$

where we have used that $S(\Lambda_{\omega_p,-\vec{p}}) = S(\Lambda_p^{-1}) = S^{-1}(\Lambda_p)$ and that $S^\dagger(\Lambda_p) = S(\Lambda_p)$, since Λ_p is a pure boost.

The general solution to the Dirac equation can therefore be written as

$$\psi(x) = \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\omega_p}\{b_s(\omega_p,\vec{p})\psi_{sp}^+(x) + d_t(\omega_p,\vec{p})\psi_{tp}^-(x)\}, \tag{1.3.65}$$

with $b_s(\omega_p,\vec{p}), d_t(\omega_p,\vec{p}) \in L^2(\mathbb{R}^3, \frac{d^3p}{\omega_p})$. The scalar product can also be written as

$$(\psi(x),\varphi(x)) = \sum_s \int \frac{1}{2\omega_p}\{\bar{b}_{s\psi}(\omega_p,\vec{p})b_{s\varphi}(\omega_p,\vec{p}) + \bar{d}_{s\psi}(\omega_p,\vec{p})d_{s\varphi}(\omega_p,\vec{p})\}. \tag{1.3.66}$$

Note that the indices s and t are not spin-indices. To establish how the Poincaré-group acts on the Hilbert-space, note that to every Lorentz transformation Λ there exists a unique pure boost Λ_p and a unique rotation R, such that

$$\Lambda = \Lambda_p R = R\Lambda_{R^{-1}p} \ \text{[20]}. \tag{1.3.67}$$

Λ_p is of course the boost taking m^μ to $\Lambda^\nu_\mu m^\nu \stackrel{\text{def}}{=} p^\mu$ and the rotation R is given by

$R^\mu_\nu = \Lambda_p^{-1\mu}{}_\alpha \Lambda^\alpha_\nu$. We therefore have

$$\begin{aligned}(U(a,\Lambda)\psi^+_{sp})(x) =& S(\Lambda)\psi^+_{sp}(\Lambda^{-1}(x-a))\\ =& S(\Lambda)S(\Lambda_p)\chi^+_s(m)e^{-ip\Lambda^{-1}(x-a)}\\ =& S(\Lambda\Lambda_p)\chi^+_s(m)e^{-ikx}e^{ika}\\ =& S(\Lambda_k R)\chi^+_s(m)e^{-ikx}e^{ika}\\ =& S(\Lambda_k)S(R)\chi^+_s(m)e^{-ikx}e^{ika}\\ =& e^{ika}\sum_t A(R)_{ts}\psi^+_{kt}(x).\end{aligned} \tag{1.3.68}$$

Here $k = \Lambda p$, $R = \Lambda_k^{-1}\Lambda\Lambda_p$, and $A(R)$ is its $SL(2,\mathbb{C})$-representant, which is unitary, since R is a rotation. Here we write $A(R)_{ts}$ for $A(R)_t{}^s$. Of course one also has

$$(U(a,\Lambda)\psi^-_{sp})(x) = e^{-ika}\sum_t A(R)_{ts}\psi^-_{kt}(x). \tag{1.3.69}$$

As a consequence

$$\begin{aligned}(U(a,\Lambda)b_t)(p) =& e^{-ipa}A(\Lambda_p^{-1}\Lambda\Lambda_{\Lambda^{-1}p})_{ts}b_s(\Lambda^{-1}p)\\ (U(a,\Lambda)d_t)(p) =& e^{ipa}A(\Lambda_p^{-1}\Lambda\Lambda_{\Lambda^{-1}p})_{ts}d_s(\Lambda^{-1}p).\end{aligned} \tag{1.3.70}$$

In the same way as we went to quantum fields in the scalar case, we "quantize" the Dirac field by making b_s and d_t annihilation-(respectively creation-)valued distributions. In fact, this is imposed by the conditions that we want a vacuum state Ω which verifies $P_\mu\Omega = 0$ and $[P_\mu, \psi(x)] = -i\partial_\mu\psi(x)$. Applying this last commutator to the vacuum state we see that if we only want to have positive energy states for the field -and this is part of the Wightman axioms- then b_s must annihilate the vacuum and $d_t(p)$ must create a positive state with energy-impulsion p^μ. This implies that if we keep the action of the Poincaré group on the Positive energy solution, one has to modify it on the space of negative energy solutions. Instead of 1.3.70 we define

$$
\begin{aligned}
(U(a,\Lambda)u)(t,p) =& e^{-ipa}A(\Lambda_p^{-1}\Lambda\Lambda_{\Lambda^{-1}p})_{ts}u(s,\Lambda^{-1}p) \\
(U(a,\Lambda)v)(t,p) =& e^{-ipa}A(\Lambda_{\Lambda^{-1}p}^{-1}\Lambda^{-1}\Lambda_p)_{st}v(s,\Lambda^{-1}p).
\end{aligned}
\tag{1.3.71}
$$

The scalar product is still Poincré invariant and we can build the antisymmetrical tensor product $\mathcal{A}(\mathcal{H}^{\otimes n})$ and the full Hilbert space $\mathcal{H} \overset{\text{def}}{=} \oplus_l \mathcal{A}(\mathcal{H}_1^{\otimes l})$ on which the creation and annihilation operators act as follows:

$$
\begin{aligned}
(b_s(p)u_n\otimes v_m)(s_1,p_1,\dots,s_{n-1},p_{n-1};t_1,q_1,\dots,t_m,q_m) =& \sqrt{n}u_n\otimes v_m(s,p,s_1,p_1,\dots,s_{n-1},p_{n-1}; \\
& t_1,q_1,\dots,t_m,q_m), \\
(b_s^\dagger(p)u_n\otimes v_m)(s_1,p_1,\dots,s_{n+1},p_{n+1};t_1,q_1,\dots,t_m,q_m) =& \frac{2\omega_p}{\sqrt{n+1}}\sum_{i=1}^{n+1}(-1)^{i+1}\delta^3(\vec{p}-\vec{p}_i)\delta_{ss_i} \\
& \times u_n\otimes v_m(s_1,p_1,\dots,\hat{s}_i,\hat{p}_i,\dots,s_{n+1},p_{n+1}; \\
& t_1,q_1,\dots,t_m,q_m); \\
(d_t(q)u_n\otimes v_m)(s_1,p_1,\dots,s_n,p_n;t_1,q_1,\dots,t_{m-1},q_{m-1}) =& (-1)^n\sqrt{m}u_n\otimes v_m(s_1,p_1,\dots,s_n,p_n; \\
& t,q,t_1,q_1,\dots,t_{m-1},q_{m-1}), \\
(d_t^\dagger(q)u_n\otimes v_m)(s_1,p_1,\dots,s_n,p_n;t_1,q_1,\dots,t_{m+1},q_{m+1}) =& \frac{2\omega_q}{\sqrt{m+1}}\sum_{i=1}^{n+1}(-1)^{i+1+n}\delta^3(\vec{q}-\vec{q}_i)\delta_{tt_i} \\
& \times u_n\otimes v_m(s_1,p_1,\dots,s_n,p_n; \\
& t_1,q_1,\dots,\hat{t}_i,\hat{q}_i,\dots,t_{m+1},q_{m+1}).
\end{aligned}
\tag{1.3.72}
$$

Again we have anti-commutation relations:

$$
\begin{aligned}
\{b_s^\sharp(p),& d_t^\sharp(q)\} = 0, \\
\{b_s(p),b_t(q)\} =& \{d_s(p),d_t(q)\} = 0, \\
\{b_s^\dagger(p),b_t^\dagger(q)\} =& \{d_s^\dagger(p),d_t^\dagger(q)\} = 0, \\
\{b_s^\dagger(p),b_t(q)\} =& \{d_s^\dagger(p),d_t(q)\} = 2\omega(p)\delta_{st}\delta^3(\vec{p}-\vec{q}).
\end{aligned}
\tag{1.3.73}
$$

The unitary representation of the Poincaré group also lifts and gives:

$$
\begin{aligned}
U(a,\Lambda)b_s(p)U^{-1}(a,\Lambda) =& A(\Lambda_p^{-1}\Lambda^{-1}\Lambda_{\Lambda p})_{st}b_t(\Lambda p)e^{-i(\Lambda p)a}\\
U(a,\Lambda)b_s^\dagger(p)U^{-1}(a,\Lambda) =& A(\Lambda_{\Lambda p}^{-1}\Lambda\Lambda_p)_{ts}b_t^\dagger(\Lambda p)e^{-i(\Lambda p)a}\\
U(a,\Lambda)d_s^\dagger(p)U^{-1}(a,\Lambda) =& A(\Lambda_p^{-1}\Lambda^{-1}\Lambda_{\Lambda p})_{st}d_t^\dagger(\Lambda p)e^{i(\Lambda p)a}\\
U(a,\Lambda)d_s(p)U^{-1}(a,\Lambda) =& A(\Lambda_{\Lambda p}^{-1}\Lambda\Lambda_p)_{ts}d_t(\Lambda p)e^{-i(\Lambda p)a}
\end{aligned}
\tag{1.3.74}
$$

All in all we have

Definition 1.3.1. The quantized Dirac field is given by

$$\psi(x) = \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\omega_p}\{b_s(\omega_p,\vec{p})\psi^+_{sp}(x) + d_t^\dagger(\omega_p,\vec{p})\psi^-_{tp}(x)\}. \tag{1.3.75}$$

Under the previous definition of $U(a,\Lambda)$ it satisfies

$$S(A)U(a,\Lambda)\psi(x)U^{-1}(a,\Lambda) = \psi(\Lambda x + a). \tag{1.3.76}$$

Now that we know the shape of a Dirac field, we are going to use it build the Majorana quantum field. First note that if $\varphi_a(x)$ is a Majorana field, then the quantity

$$\psi^M(x) \overset{\text{def}}{=} \begin{pmatrix}\varphi_a(x)\\ \bar{\varphi}^{\bar{b}}(x)\end{pmatrix} \tag{1.3.77}$$

is a solution to the Dirac equation and must be of the form 1.3.75. Here $\bar{\varphi}^{\bar{b}}(x) = (\varphi^b)^\dagger(x)$. In addition, the spinor field $\psi^M(x)$ satisfies

$$\psi^M(x) = \begin{pmatrix}-i\sigma_2 & 0\\ 0 & i\sigma_2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(\psi^M(x))^{\dagger T} \overset{\text{def}}{=} C\gamma^0\psi^M(x)^{\dagger T}, \tag{1.3.78}$$

since, in our representation, $-i\sigma_2 = \epsilon_{ab} = -\epsilon^{\bar{a}\bar{b}}$. Conversely, a Dirac spinor obeying 1.3.78 is easily seen to be made of one Majorana field.

Putting all together and after some computations, we obtain the following

Theorem 1.3.2. *The quantum spinor $\psi^M(x)$ is equal to*

$$\psi^M(x) = \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\omega_p}\{b_s(\vec{p})\psi^+_{sp}(x) + b^\dagger_{\neg s}(\vec{p})(-1)^s\psi^-_{sp}(x)\}, \tag{1.3.79}$$

where $\neg s = 3 - s$. As a consequence, a Majorana spinor field $\varphi_a(x)$ can be written as

$$\varphi_a(x) = \frac{\sqrt{m}}{(2\pi)^2}\int \frac{d^3p}{2\omega_p}\{b_s(\vec{p})e^{-ipx} + b^\dagger_{\neg s}(\vec{p})(-1)^s e^{ipx}\}A(\Lambda_p)_{as}, \tag{1.3.80}$$

with $A(\Lambda_p)$ the $SL(2,\mathbb{C})$-representant of the Lorentz-boost Λ_p.

The various anticommutation relations of the field become:

$$\begin{aligned}\{\varphi_a(x), \varphi_b(y)\} =& im\epsilon_{ab}D_m(x-y),\\ \{\varphi_a(x), \bar{\varphi}_{\bar{b}}(y)\} =& \sigma^\mu_{a\bar{b}}\partial_\mu D_m(x-y).\end{aligned} \tag{1.3.81}$$

1.3.5 Majorana Ghost Fields

The anticommutation relations for the spinor fields used in the last subsection are forced by the already encountered spin and statistics theorem, which, in the spinor case, takes the following form:

Theorem 1.3.3. *If a spinor field $\varphi_a(x)$ satisfies causal commutation relations, i.e.*

$$\begin{aligned}[\varphi_a(x), \varphi_b(y)] =& 0,\\ [\varphi_a(x), \bar{\varphi}_{\bar{b}}(y)] =& 0, \ \textit{if } x \sim y,\end{aligned} \tag{1.3.82}$$

then $\varphi(x) = 0$.

Proof. A proof can be found in [27] □

Here and as usual $\bar{\varphi}_{\bar{b}}(y) = \varphi^\dagger_b(y)$.

However, As for the complex scalar field, one may also define quantum ghost and anti-ghost Majorana fields $\chi_a(x)$ and $\tilde{\chi}_a(x)$, having unphysical commutation relations. Let's define the following spinor field

$$\chi_a(x) \stackrel{\text{def}}{=} \frac{\sqrt{m}}{(2\pi)^2} \int \frac{d^3p}{2\omega_p} \{b_s(\vec{p})e^{-ipx} + d^\dagger_{\neg s}(\vec{p})(-1)^s e^{ipx}\} A(\Lambda_p)_{as}, \tag{1.3.83}$$

with $b_s(\vec{p})$ and $d_s(\vec{p})$ being annihilation-operators acting on two different Hilbert spaces and having the commutation relations

$$\begin{aligned}
[b_s(\vec{p}), b^\dagger_t(\vec{q})] =& 2\omega_p \delta_{st} \delta^3(\vec{p}-\vec{q}) = [d_s(\vec{p}), d^\dagger_t(\vec{q})], \\
[b_s(\vec{p}), b_t(\vec{q})] =& 0 = [d_s(\vec{p}), d_t(\vec{q})], \\
[b^\dagger_s(\vec{p}), d_t(\vec{q})] =& 0 = [b_s(\vec{p}), d^\dagger_t(\vec{q})].
\end{aligned} \tag{1.3.84}$$

The action of these operators can be achieved on a symmetrized Hilbert-space similar to the one constructed for the Dirac-field by omitting the various minus-signs in 1.3.72 . The unitary representation of the Poincaré-group is like the one used for the "physical" Majorana-field.

On this Hilbert-space we define now a conjugation K which is an antilinear involution:

$$ab_s(\vec{p})^K \stackrel{\text{def}}{=} a^* d^\dagger_s(\vec{p}), \quad a \in \mathbb{C}. \tag{1.3.85}$$

This involution behaves similarly to the usual adjoint operation. For the ghost quantum Majorana-field $\chi_a(x)$ we therefore write

$$\bar{\chi}_{\bar{b}} \stackrel{\text{def}}{=} \frac{-i}{m} \sigma^\mu_{a\bar{b}} \partial_\mu \chi^a(x) = (\chi_b)^K(x) \neq (\chi_b)^\dagger(x) \tag{1.3.86}$$

and we recover the usual Majorana-field equation as well as the commutation-relations

$$[\chi_a(x), \chi_b(y)] = [\chi_a(x), \bar{\chi}_{\bar{b}}(y)] = 0. \tag{1.3.87}$$

As in the scalar case we also define an anti-ghost Majorana-field

$$\tilde{\chi}_a(x) \stackrel{\text{def}}{=} \frac{\sqrt{m}}{(2\pi)^2} \int \frac{d^3p}{2\omega_p} \{-d_s(\vec{p})e^{-ipx} + b^\dagger_{\neg s}(\vec{p})(-1)^s e^{ipx}\} A(\Lambda_p)_{as}. \tag{1.3.88}$$

But now one has

$$\bar{\tilde{\chi}}_{\bar{b}} \stackrel{\text{def}}{=} \frac{-i}{m}\sigma^{\mu}_{a\bar{b}}\partial_{\mu}\tilde{\chi}^{a}(x) = -(\tilde{\chi}_{b})^{K}(x). \tag{1.3.89}$$

This minus sign will be important when we will study the gauge-variation of the anti-ghost super-field.

The various commutation relations read:

$$\begin{aligned}
[\chi_a(x), \tilde{\chi}_b(y)] =& im\epsilon_{ab}D_m(x-y), \\
[\bar{\chi}_{\bar{a}}(x), \tilde{\chi}_b(y)] =& \sigma^{\mu}_{\bar{a}b}\partial_{\mu}D_m(x-y), \\
[\tilde{\chi}_a(x), \tilde{\chi}_b(y)] =& 0 = [\bar{\tilde{\chi}}_{\bar{a}}(x), \tilde{\chi}_b(y)].
\end{aligned} \tag{1.3.90}$$

1.3.6 Vector Fields

Free massive vector fields are fields satisfying the equation

$$(\Box + m^2)A^{\mu}(x) = 0. \tag{1.3.91}$$

Their general classical solutions are

$$A^{\mu}(x) = \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\omega(p)}(a^{\mu}(\vec{p})e^{-ip_{\nu}x^{\nu}} + a^{\mu *}(\vec{p})e^{ip_{\nu}x^{\nu}}), \tag{1.3.92}$$

with $p_0 = \sqrt{\vec{p}^2 + m^2}$.

As in the previous cases we expect the $a^{\mu}(\vec{p})$ and $a^{*\mu}(\vec{p})$ to become operator valued distributions, having as commutation relations:

$$[a^{\mu}(\vec{p}), a^{\nu\dagger}(\vec{q})] = -\eta^{\mu\nu}2\omega(p)\delta^3(\vec{p}-\vec{q}). \tag{1.3.93}$$

The tensor $\eta^{\mu\nu}$ must be present for reasons of relativistic covariance. But this then implies that the Fock-space constructed has an indefinite scalar product:

$$\begin{aligned}\|a^{0\dagger}(f)\|^2 &= \int \frac{d^3p}{2\omega(p)}\frac{d^3q}{2\omega(q)} f^*(\vec{q})f(\vec{p})(a^{0\dagger}(\vec{p})\Omega, a^{0\dagger}(\vec{q})\Omega) \\ &= \int \frac{d^3p}{2\omega(p)}\frac{d^3q}{2\omega(q)} f^*(\vec{q})f(\vec{p})(\Omega, [a^0(\vec{p}), a^{0\dagger}(\vec{q})]\Omega) \\ &= -\int \frac{d^3p}{2\omega(p)}\frac{d^3q}{2\omega(q)} f^*(\vec{q})f(\vec{p})(\Omega, \eta^{00}2\omega(p)\delta^3(\vec{p}-\vec{q})\Omega) \\ &= -\int \frac{d^3p}{2\omega(p)} f^*(\vec{p})f(\vec{p})(\Omega,\Omega) < 0.\end{aligned} \tag{1.3.94}$$

One is therefore forced to consider a physical subspace of the Fock-space. This physical subspace should satisfy the first Wightman axiom. To make this choice more transparent we rewrite 1.3.92 as

$$A^\mu(x) = \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\omega(p)}(a^s(\vec{p})\epsilon_s^\mu(\vec{p})e^{-ip_\nu x^\nu} + a^{t\dagger}(\vec{p})\epsilon_t^\mu(\vec{p})e^{ip_\nu x^\nu}), \tag{1.3.95}$$

where

$$\epsilon_s^\mu(\vec{p}) = \Lambda(\vec{p})_t^\mu \delta_s^t, \tag{1.3.96}$$

with $\Lambda(\vec{p})_t^\mu$ the pure boost which takes $(m,0,0,0)$ to (p^μ). One easily verifies

$$\epsilon_s^\mu(\vec{p})\epsilon_{t\mu}(\vec{p}) = \eta_{st}. \tag{1.3.97}$$

One then has

$$a^{t\sharp}(\vec{p}) = \epsilon^{t\mu}(\vec{p})a_\mu^\sharp(\vec{p}), \tag{1.3.98}$$

with commutation relations

$$\begin{aligned}[a^t(\vec{p}), a^{s\dagger}(\vec{q})] =& \epsilon^{t\mu}(\vec{p})\epsilon^{s\nu}(\vec{q})[a_\mu(\vec{p}), a_\nu^\dagger(\vec{q})] \\ =& -\epsilon^{t\mu}(\vec{p})\epsilon_\mu^s(\vec{q})2\omega(p)\delta^3(\vec{p}-\vec{q}) \\ =& -2\eta^{st}\omega(p)\delta^3(\vec{p}-\vec{q}).\end{aligned} \tag{1.3.99}$$

The advantage of this construction is the manifestly covariant splitting between the subspaces with negative and positive defined scalar product. Indeed, as is surely known by the reader, a proper orthochronous Lorentz transformation Λ can be uniquely decomposed into a rotation-boost product $\Lambda = R\Lambda(\vec{v})$ as well as a boost-rotation product $\Lambda = \Lambda(R\vec{v})R$ [20]. One then has

$$\begin{aligned}\Lambda^\mu_\nu \epsilon^\nu_t(\vec{p}) &\stackrel{1.3.96}{=} \Lambda^\mu_\nu \Lambda^\nu_t(\vec{p}) \stackrel{\text{def}}{=} \tilde{\Lambda}^\mu_t \\ &= \Lambda(R\vec{v})^\mu_\nu R^\nu_t,\end{aligned} \tag{1.3.100}$$

for well chosen R and $\vec{v}$. By setting $t = 0$ one must have

$$\begin{aligned}\Lambda^\mu_\nu \epsilon^\nu_0(\vec{p}) &= \Lambda^\mu_\nu p^\nu \\ &= \Lambda(R\vec{v})^\mu_\nu R^\nu_0 = \Lambda(R\vec{v})^\mu_\nu \delta^\nu_0 \\ &= \Lambda(R\vec{v})^\mu_0,\end{aligned} \tag{1.3.101}$$

showing that one must choose $R\vec{v} = \vec{\Lambda p}$. As a result we get

$$\Lambda^\mu_\nu \epsilon^\nu_t(\vec{p}) = R^s_t \epsilon^\mu_s(\vec{\Lambda p}), \tag{1.3.102}$$

where R depends on p^μ and Λ. To find out how the Poincarre representation acts on the annihilation operators we start from the following requirement:

$$\Lambda^\nu_\mu U(\Lambda, a) a^s(\vec{p}) \epsilon^\mu_s(\vec{p}) U^{-1}(\Lambda, a) = e^{-i\langle \Lambda p, a\rangle} \epsilon^\nu_s(\vec{\Lambda p}) a^s(\vec{\Lambda p}) \tag{1.3.103}$$

(To compare with 1.3.13). This equation implies:

$$\begin{aligned}\Lambda^\nu_\mu U(\Lambda, a) a^s(\vec{p}) \epsilon^\mu_s(\vec{p}) U^{-1}(\Lambda, a) =& \Lambda^\nu_\mu \epsilon^\mu_s(\vec{p}) U(\Lambda, a) a^s(\vec{p}) U^{-1}(\Lambda, a) \\ \stackrel{1.3.102}{=}& R^t_s \epsilon^\nu_t(\vec{\Lambda p}) U(\Lambda, a) a^s(\vec{p}) U^{-1}(\Lambda, a) \\ \stackrel{1.3.103}{\Rightarrow} R^t_s U(\Lambda, a) a^s(\vec{p}) U^{-1}(\Lambda, a) =& e^{-i\langle \Lambda p, a\rangle} a^t(\vec{\Lambda p}), \\ R^t_s U(\Lambda, a) a^{s\dagger}(\vec{p}) U^{-1}(\Lambda, a) =& e^{i\langle \Lambda p, a\rangle} a^{t\dagger}(\vec{\Lambda p}).\end{aligned} \tag{1.3.104}$$

This shows that under a Poincaré transformation, and since R_s^t is a rotation, the operators $a^{0\sharp}(\vec{p})$ do not mix with the "positive metric" operators $a^{i\sharp}(\vec{p}), \quad i = 1, 2, 3$. Therefore one can define the hilbert space $\mathcal{H}_{\text{phys}}$ of physical states to be the Fock-space generated from the vacuum Ω by the operators $a^{i\sharp}(\vec{p}), \quad i = 1, 2, 3$. This Fock-space is relativistically invariant and carries the clearly unitary representation $U(\Lambda, a)$.

One can also define a conjugation operation $a_s(\vec{p}) \mapsto a_s^K(\vec{p})$ by imposing

$$\begin{aligned} a^{0K}(\vec{p}) =& - a^{0\dagger}(\vec{p}), \\ a^{iK}(\vec{p}) =& a^{i\dagger}(\vec{p}), \quad i = 1, 2, 3. \end{aligned} \tag{1.3.105}$$

This definition is independant of the choice of coordinates.

With the help of this conjugation we redefine the vector field as

$$A^\mu(x) = \frac{1}{(2\pi)^2} \int \frac{d^3p}{2\omega(p)} (a^s(\vec{p}) \epsilon_s^\mu(\vec{p}) e^{-ip_\nu x^\nu} + a^{tK}(\vec{p}) \epsilon_t^\mu(\vec{p}) e^{ip_\nu x^\nu}), \tag{1.3.106}$$

This field is now self-conjugate and is required to satisfy the commutation relations

$$[A^\mu(x), A^\mu(y)] = i\eta^{\mu\nu} D_m(x - y), \tag{1.3.107}$$

which now implies

$$[a^t(\vec{p}), a^{sK}(\vec{q})] = -\eta^{st}\omega(p)\delta^3(\vec{p} - \vec{q}). \tag{1.3.108}$$

The operators $a^t(\vec{p})$ and $a^{s\dagger}(\vec{q})$ now generate a positive definite Hilbert space $\mathcal{H}$, but the Poincaré representation is only unitary when restricted to the physical subspace $\mathcal{H}_{\text{phys}}$, both beeing defined as before.

The quantity

$$\begin{aligned} i\partial_\mu A^\mu(x) =& \frac{1}{(2\pi)^2} \int \frac{d^3p}{2\omega(p)} (p_\mu a^s(\vec{p}) \epsilon_s^\mu(\vec{p}) e^{-ip_\nu x^\nu} - p_\mu a^{tK}(\vec{p}) \epsilon_t^\mu(\vec{p}) e^{ip_\nu x^\nu}) \\ =& \frac{m}{(2\pi)^2} \int \frac{d^3p}{2\omega(p)} (a^0(\vec{p}) e^{-ip_\nu x^\nu} + a^{0\dagger}(\vec{p}) e^{ip_\nu x^\nu}) \end{aligned} \tag{1.3.109}$$

has a vanishing mean value on any physical state. Moreover, the kernel of the positive frequency part of 1.3.109 defines the physical subspace of the vector field. This is precisely the place where we can make use of the scalar ghost fields.

We will use here a scalar ghost field $u(x)$:

$$\begin{aligned}
u(x) =& \frac{1}{(2\pi)^2}\int \frac{d^3k}{2\omega(k)}(b(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + c^\dagger(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)})\\
:=& \frac{1}{(2\pi)^2}\int \frac{d^3k}{2\omega(k)}((c^\dagger)^K(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + b^K(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)}),\\
\tilde{u}(x) =& \frac{1}{(2\pi)^2}\int \frac{d^3k}{2\omega(k)}(-c(\vec{k})e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} - b^K(\vec{k})e^{(i\langle(\omega(k),\vec{k}),x\rangle)}).
\end{aligned} \tag{1.3.110}$$

Here we have defined a conjugation operator K, so that $u(x)$ becomes self-conjugate. We also have:

$$\begin{aligned}
(bc)^K =& c^K b^K,\\
(\lambda b)^K =& \lambda^* b^K, \quad \lambda \in \mathbb{C}.
\end{aligned} \tag{1.3.111}$$

The anticommutation relations are:

$$\begin{aligned}
\{u(x), \tilde{u}(y)\} =& -iD_m(x-y),\\
\{u(x), u(y)\} =& 0.
\end{aligned} \tag{1.3.112}$$

This allows one to define the so-called gauge charge Q:

$$
\begin{aligned}
Q &\stackrel{\text{def}}{=} \int d^3x \partial_\mu A^\mu(x) \overleftrightarrow{\partial_0} u(x) \\
&= \int d^3x \frac{-im}{(2\pi)^2} \int \frac{d^3p}{2\omega(p)} (a^0(\vec{p}) e^{-ip_\nu x^\nu} + a^{0\dagger}(\vec{p}) e^{ip_\nu x^\nu}) \times \\
&\quad \overleftrightarrow{\partial_0} \frac{1}{(2\pi)^2} \int \frac{d^3k}{2\omega(k)} (b(\vec{k}) e^{(-i\langle(\omega(k),\vec{k}),x\rangle)} + c^\dagger(\vec{k}) e^{(i\langle(\omega(k),\vec{k}),x\rangle)}) \\
&= \frac{-im}{(2\pi)^4} \int d^3x \frac{d^3p}{2\omega(p)} \frac{d^3k}{2\omega(k)} \{ (ip_0 a^0(\vec{p}) e^{-i\langle p,x\rangle} - ip_0 a^{0\dagger}(\vec{p}) e^{i\langle p,x\rangle}) \times \\
&\quad (b(\vec{k}) e^{-i\langle p,x\rangle} + c^\dagger(\vec{p}) e^{i\langle p,x\rangle}) + (a^0(\vec{p}) e^{-i\langle p,x\rangle} + a^{0\dagger}(\vec{p}) e^{i\langle p,x\rangle}) \times \\
&\quad (-ik_0 b(\vec{k}) e^{-i\langle p,x\rangle} + ik_0 c^\dagger(\vec{p}) e^{i\langle p,x\rangle}) \} \\
&= \frac{m}{(2\pi)^4} \int d^3x \frac{d^3p}{2\omega(p)} \frac{d^3k}{2\omega(k)} \{ p_0 a^0(\vec{p}) c^\dagger(\vec{k}) e^{-i\langle p-k,x\rangle} + k_0 a^0(\vec{p}) c^\dagger(\vec{k}) e^{-i\langle p-k,x\rangle} \\
&\quad - p_0 a^{0\dagger}(\vec{p}) b(\vec{k}) e^{+i\langle p-k,x\rangle} - k_0 a^{0\dagger}(\vec{p}) b(\vec{k}) e^{i\langle p-k,x\rangle} \} \\
&= \frac{m}{2\pi} \int \frac{d^3p}{2\omega(p)} \frac{d^3k}{2\omega(k)} \{ p_0 a^0(\vec{p}) c^\dagger(\vec{k}) e^{-i(p_0-k_o)x^0} \delta^3(\vec{p}-\vec{k}) + k_0 a^0(\vec{p}) c^\dagger(\vec{k}) e^{-i(p_0-k_o)x^0} \delta^3(\vec{p}-\vec{k}) \\
&\quad - p_0 a^{0\dagger}(\vec{p}) b(\vec{k}) e^{+i(p_0-k_0)x^0} \delta^3(\vec{p}-\vec{k}) - k_0 a^{0\dagger}(\vec{p}) b(\vec{k}) e^{i(p_0-k_0)x^0} \delta^3(\vec{p}-\vec{k}) \} \\
&= \frac{m}{2\pi} \int \frac{d^3p}{2\omega(p)} \{ a^0(\vec{p}) c^\dagger(\vec{p}) - a^{0\dagger}(\vec{p}) b(\vec{p}) \}.
\end{aligned}
\tag{1.3.113}
$$

This expression shows that Q is Poincaré invariant and self-conjugate.

Let us first study the domain of definition of $D(Q)$. Let us write $\mathcal{H} = \mathcal{H}_{\text{vec}} \otimes \mathcal{H}_{\text{ghost}}$. The vector space $\mathcal{D}_{\text{vec}} \otimes \mathcal{D}_{\text{ghost}}$ is a dense set of vectors in $\mathcal{H}$. On a typical vector of this space the gauge operator acts as:

$$
\begin{aligned}
&\Psi_{n,n_o,m_b,m_c}(\vec{k}_1,\dots,\vec{k}_n,\vec{q}_1,\dots,\vec{q}_{n_0},\vec{l}_1,\dots,\vec{l}_{n_b},\vec{j}_1,\dots,\vec{j}_{n_c}) \mapsto \\
&\Psi'_{n,n_o-1,m_b,m_c+1}(\vec{k}_1,\dots,\vec{k}_n,\vec{q}_1,\dots,\vec{q}_{n_0-1},\vec{l}_1,\dots,\vec{l}_{n_b},\vec{j}_1,\dots,\vec{j}_{n_c+1})+ \\
&\Psi'_{n,n_o+1,m_b-1,m_c}(\vec{k}_1,\dots,\vec{k}_n,\vec{q}_1,\dots,\vec{q}_{n_0+1},\vec{l}_1,\dots,\vec{l}_{n_b-1},\vec{j}_1,\dots,\vec{j}_{n_c}) \\
&= \sqrt{\frac{n_0}{m_c+1}}\sum_{i=1}^{m_c+1}(-1)^{i+1+m_b}\Psi_{n,n_o,m_b,m_c}(\vec{k}_1,\dots,\vec{k}_n,\vec{j}_i,\vec{q}_1,\dots,\vec{q}_{n_0-1},\vec{l}_1,\dots,\vec{l}_{n_b}, \\
&\qquad \vec{j}_1,\dots,\hat{\vec{j}_i},\dots,\vec{j}_{n_c+1}) \\
&-\sqrt{\frac{m_b}{n_0+1}}\sum_{r=1}^{m_0+1}\Psi_{n,n_o,m_b,m_c}(\vec{k}_1,\dots,\vec{k}_n,\vec{q}_1,\dots,\hat{\vec{q}_k},\dots,\vec{q}_{n_0+1},\vec{q}_k,\vec{l}_1,\dots,\vec{l}_{n_b-1}, \\
&\qquad \vec{j}_1,\dots,\vec{j}_{n_c}),
\end{aligned}
\tag{1.3.114}
$$

where we have used the shorthand notation Ψ_{n,n_o,m_b,m_c} for $\Psi_n^{\text{phys}}\otimes\Psi_{n_o}\otimes\Psi_{m_b}\otimes\Psi_{m_c}$ with Ψ_n^{phys} a physical vector state obtained by acting n times on the vacuum with one of the creation operators $a^l(\vec{p})$, $l=1,2,3$.

Therefore we see that Q is densely defined.

It can also be seen that if a state is physical, that is if a state has no "scalar" vector particles, i.e. no zero-vector-component state, and no ghost particles, then this state is in the Kernel of Q:

$$
\mathcal{H}_{\text{phys}} \subset Ker(Q). \tag{1.3.115}
$$

But there is one defect in this construction: Q is not nilpotent (nilpotency will be seen to be important). To include this property one will need another massive herimitian scalar field $g(x)$, called the Goldstone field. The new definition of the gauge charge Q is:

$$Q \stackrel{\text{def}}{=} \int d^3x(\partial_\mu A^\mu(x) + mg(x))\overleftrightarrow{\partial_0}u(x)$$
$$= \frac{m}{\pi}\int \frac{d^3p}{2\omega(p)}\{(a^0(\vec{p}) + m\tilde{g}(\vec{p}))c^\dagger(\vec{p}) + (-a^{0\dagger}(\vec{p}) + m\tilde{g}^\dagger(\vec{p}))b(\vec{p})\}. \tag{1.3.116}$$

Its adjoint, $Q^\dagger$, is given by

$$Q^\dagger = \frac{m}{\pi}\int \frac{d^3p}{2\omega(p)}\{(-a^0(\vec{p}) + m\tilde{g}(\vec{p}))b^\dagger(\vec{p}) + (a^{0\dagger}(\vec{p}) + m\tilde{g}^\dagger(\vec{p}))c(\vec{p})\}. \tag{1.3.117}$$

The domain $D(Q)$ and $D(Q^\dagger)$ of these newly defined gauge charge operators are also dense.

Considering the goldstone field as a physical field and the ghost fields and the scalar component of the vetor fields as unphysical, one has

$$\mathcal{H}_{\text{phys}} \subset Ker(Q)$$
$$\mathcal{H}_{\text{phys}} \subset Ker^\dagger(Q) \tag{1.3.118}$$
$$\mathcal{H}_{\text{phys}} = Ker^\dagger(Q) \cap Ker(Q).$$

The so-called gauge-variation is also defined using Q:

Definition 1.3.2. The gauge variation of $A^\mu(x)$, $d_Q A^\mu(x)$, is defined as beeing the commutator of Q with $A^\mu(x)$:

$$d_Q A^\mu(x) \stackrel{\text{def}}{=} [Q, A^\mu(x)]. \tag{1.3.119}$$

Similarly, for the goldstone field we define:

$$d_Q g(x) \stackrel{\text{def}}{=} [Q, g(x)]. \tag{1.3.120}$$

For the ghost fields, the gauge variation is defined using the anti-commutator:

$$d_Q u(x) \stackrel{\text{def}}{=} \{Q, u(x)\},$$
$$d_Q \tilde{u}(x) \stackrel{\text{def}}{=} \{Q, \tilde{u}(x)\}. \tag{1.3.121}$$

For our vector field one can easily see that:

$$\begin{aligned} d_Q A^\nu(x) =&[Q, A^\nu(x)] = \int d^3y \partial^y_\mu [A^\mu(y) \overleftrightarrow{\partial}^y_0 u(y), A^\nu(x)] \\ =& \int d^3y \partial^y_\mu [A^\mu(y), A^\nu(x)] \overleftrightarrow{\partial}^y_0 u(y) \\ =& i \int d^3y \partial^y_\mu \eta^{\mu\nu} D_m(y-x) \overleftrightarrow{\partial}^y_0 u(y) \\ \overset{1.3.26}{=}& \frac{i}{2\pi} \partial^\nu_x u(x). \end{aligned} \tag{1.3.122}$$

The same computation gives for the goldstone field:

$$\begin{aligned} d_Q g(x) =&[Q, g(x)] = \int d^3y [m g(y) \overleftrightarrow{\partial}^y_0 u(y), g(x)] \\ =& \int d^3y [m g(y), g(x)] \overleftrightarrow{\partial}^y_0 u(y) \\ =& - i \int d^3y m D_m(y-x) \overleftrightarrow{\partial}^y_0 u(y) \\ \overset{1.3.26}{=}& \frac{im}{2\pi} u(x). \end{aligned} \tag{1.3.123}$$

Similarly, for the ghost fields, one has

$$\begin{aligned} d_Q u(x) =&\{Q, u(x)\} = \int d^3y \{(\partial^y_\mu A^\mu(y) + m g(y)) \overleftrightarrow{\partial}^y_0 u(y), u(x)\} \\ =& \int d^3y (\partial^y_\mu A^\mu(y) + m g(y)) \overleftrightarrow{\partial}^y_0 \{u(y), u(x)\}; \\ =&0; \\ d_Q \tilde{u}(x) =&\{Q, \tilde{u}(x)\} = \int d^3y \{(\partial^y_\mu A^\mu(y) + m g(y)) \overleftrightarrow{\partial}^y_0 u(y), \tilde{u}(x)\} \\ =& \int d^3y (\partial^y_\mu A^\mu(y) + m g(y)) \overleftrightarrow{\partial}^y_0 \{u(y), \tilde{u}(x)\} \\ =& - i \int d^3y (\partial^y_\mu A^\mu(y) + m g(y)) \overleftrightarrow{\partial}^y_0 D_m(y-x) \\ =& \frac{-i}{2\pi} (\partial_\mu A^\mu(x) + m g(x)). \end{aligned} \tag{1.3.124}$$

The nilpotency of Q follows from these gauge variations:

$$\begin{aligned}Q^2 =&\frac{1}{2}\{Q,Q\} = \frac{1}{2}\int d^3x\{(\partial_\mu A^\mu(x)+mg(x))\overleftrightarrow{\partial_0}u(x),Q\}\\ =&\frac{1}{2}\int d^3x(\partial_\mu A^\mu(x)+mg(x))\overleftrightarrow{\partial_0}\{u(x),Q\}+\frac{1}{2}\int d^3x[Q,\partial_\mu A^\mu(x)+mg(x)]\overleftrightarrow{\partial_0}u(x)\\ =&0+\frac{i}{2}\int d^3x[\frac{1}{2\pi}\Box u(x)+\frac{m^2}{2\pi}u(x)]\overleftrightarrow{\partial_0}u(x)\\ =&0.\end{aligned} \tag{1.3.125}$$

Now we will use the properties of Q to give a uesfull caracterisation of the physical Hilbert space. The equation

$$(Q\Psi,\Phi)=(\Psi,Q^\dagger\Phi) \tag{1.3.126}$$

shows that $Ker(Q)$ is orthogonal to $Ran(Q^\dagger)$ (This is true for any operator). In fact we have

$$\mathcal{H}=Ker(Q)\oplus\overline{Ran(Q^\dagger)}=Ker(Q^\dagger)\oplus\overline{Ran(Q)}. \tag{1.3.127}$$

Indeed, suppose $\Psi\in D(Q)$ and $\Psi\perp\overline{Ran(Q^\dagger)}$. Then we must have $(\Psi,Q^\dagger\Phi)=0$ $\forall\Phi\in D(Q^\dagger)$, and $(Q\Psi,\Phi)=0$, so, since $D(Q)$ and $D(Q^\dagger)$ are dense in $\mathcal{H}$, we can conclude that $\Psi\in Ker(Q)$, proving the statement 1.3.127.

For the following we need the nilpotency of Q. Suppose $\Psi\in\mathcal{H}_{\text{phys}}$. Then we know that $\Psi\in Ker(Q)\cap Ker(Q^\dagger)$. Therefore we have $0=(Q\Psi,\Phi)=(\Psi,Q^\dagger\Phi)$ $\forall\Phi\in D(Q^\dagger)$ and so $\Psi\perp\overline{Ran(Q^\dagger)}$. Similarly we prove $\Psi\perp\overline{Ran(Q)}$. $Q^2=0$ implies $0=(\Psi,Q^2\Phi)=(Q^\dagger\Psi,Q\Phi)$, $\forall\Psi\in D(Q^\dagger)$ and $\forall\Phi\in D(Q)$, and therefore $\overline{Ran(Q^\dagger)}\perp\overline{Ran(Q)}$. Moreover, if $\Psi\in Ker(Q)$ and $Q\perp\overline{Ran(Q)}$ then $\forall\Phi\in D(Q)$ $0=(\Psi,Q\Phi)=(Q^\dagger\Psi,\Phi)$, which implies $\Psi\in Ker(Q^\dagger)$ and therefore $\Psi\in\mathcal{H}_{\text{phys}}$. This leads us to the decomposition

$$\mathcal{H}=\mathcal{H}_{\text{phys}}\oplus\overline{Ran(Q)}\oplus\overline{Ran(Q^\dagger)}, \tag{1.3.128}$$

or, equivalentely,

$$\begin{aligned}\mathcal{H}_{\text{phys}} =& Ker(Q) \ominus \overline{Ran(Q)} \equiv Ker(Q)/\overline{Ran(Q)} \\ =& Ker(Q^\dagger) \ominus \overline{Ran(Q^\dagger)} \equiv Ker(Q^\dagger)/\overline{Ran(Q^\dagger)}.\end{aligned} \tag{1.3.129}$$

1.4 The Reconstruction Theorem

After these examples of how to construct free quantum fields, one could wonder if there are different, i.e. unitary unequivalent, constructions. This section intends to show that actually there aren't many ways of doing so.

Let $\varphi_{i_j}(x)$ be a family of quantum fields, in the sense that they all satisfy the Wightman axioms defined in the first subsection. The various subscripts i_j label different kinds of spinorial tensor fields appearing in this family. Let us now consider the vacuum expectation values of all possible polynomials in the fields smeared out with test functions $f_j \in \mathcal{S}(\mathbb{R}^4)$:

$$(\Omega, \varphi_{i_1}(f_1) \dots \varphi_{i_n}(f_n)\Omega). \tag{1.4.1}$$

These quantities are all perfectly well defined since

$$\prod_{k=1}^{n} \varphi_{i_k}(f_k)\Omega \in \mathcal{D}. \tag{1.4.2}$$

One can rewrite 1.4.1 also as

$$\underbrace{\int \dots \int}_{n} W^n_{i_1 \dots i_n}(x^1, \dots, x^n) \prod_{j=1}^{n} f_j(x^j) d^4x^1 \dots d^4x^n, \tag{1.4.3}$$

where

$$W^n_{i_1 \dots i_n}(x^1, \dots, x^n) \stackrel{\text{def}}{=} (\Omega, \varphi_{i_1}(x^1) \dots \varphi_{i_n}(x^n)\Omega). \tag{1.4.4}$$

These objects are called the Wightman-functions. They can be linearly extended by continuity by virtue of the Schwartz Nuclear theorem to distributions defined on the space $\mathcal{S}(\mathbb{R}^{4n})$.

The Reconstruction Theorem now states that once all the Wightman-functions are given for all $n \in \mathbb{N}$, then there exists a unique way, up to unitary equivalence, to reconstruct the whole Hilbert-space $\mathcal{H}$, comprising the vacuum-state Ω, and the actions of the various field-operators on it. We owe this result to the so-called GNS-construction (named so after the authors Gelfand, Naimark and Segal [24], [13]) and its adaptation to the quantum fields by Wightman [30].

Let us study the simple case where the family of quantum fields consists of one free scalar hermitian field $\varphi(x)$. We have the following

Theorem 1.4.1. *a) For n odd we have*

$$W(x_1, \ldots, x_n) = 0. \tag{1.4.5}$$

b) For n even we have the "cluster expansion"

$$W(x_1, \ldots, x_n) = \sum \prod_k W(x_{i_k}, x_{j_k}), \tag{1.4.6}$$

where the sum runs over all possible partitions of the set of variables into $\frac{n}{2}$ pairs (x_{i_k}, x_{j_k}) with $i_k < j_k$.

Proof. A proof can be found in [25] or in [27]. □

Therefore, once we know the 2-point $W(x, y)$, we can uniquely reconstruct the field up to a unitary equivalence.

One even can go further: Because of the translational invariance of the field one has $W(x, y) = W(x - y) := w(\xi)$. Taking the Fourier transform on both sides one arrives at

$$\tilde{W}(p_1, p_2) = \delta^4(p_1 + p_2)\tilde{w}(p_1), \tag{1.4.7}$$

with the Klein-Gordon equation imposing

$$(P^2 - m^2)\tilde{w}(p) = 0 \Rightarrow \tilde{w}(p) = b\Theta(p^0)\delta(p^2 - m^2). \tag{1.4.8}$$

Transforming back into x-space we get

$$W(x_1, x_2) = \frac{b}{(2\pi)^2} \int \frac{d^3p}{2\omega_p} e^{-ip(x-y)} \stackrel{\text{def}}{=} -ib\Delta_+(x-y). \tag{1.4.9}$$

Note that

$$\Delta_+(x-y) - \Delta_+(y-x) = D_m(x-y) = \frac{i}{b} W([x,y]) = \frac{i}{b}(\Omega, [\varphi(x), \varphi(y)]\Omega). \tag{1.4.10}$$

Remarkably enough, theorem 1.4.1 implies that

$$W(\ldots, [x,y], \ldots) = W(\ldots)W([x,y]), \tag{1.4.11}$$

which implies that, as an operator-valued distribution equality, one has

$$[\varphi(x), \varphi(y)] = -\frac{i}{b} D_m(x-y)\mathbb{1}, \tag{1.4.12}$$

which is a $\mathbb{C}$-number (depending in this case on the value of b), a specificity for all free fields!

Summing up we know now that for a scalar free field once the commutator $[\varphi(x), \varphi(y)]$ is given, the whole field is determined, up to a unitary equivalence. This discussion is valid in general for any type of tensorial or spinorial free quantum fields. Therefore one actually has only one choice for an explicit construction of a free quantum field.

Chapter 2

Axiomatic scattering theory and the Epstein-Glaser method

2.1 Introduction

Suppose one has a field $\varphi(x)$ satisfying the Wightman axioms. Free fields are easy to build and one knows explicit solution for them. But for the interacting case things are much more complicated. In general, a mathematical rigorous definition of the equation of motion is itself dependent of its solution. So in this case one has to find the solution to an equation and define this very equation at the same time! This can be carried through perturbatively to a certain satisfactorily level by the so-called renormalization program [25].

If one has $1+1$ or $2+1$ space-time dimensions then explicit constructions exist for an interacting theory [26], but in the the physical desirable case of a $3+1$ dimensional space-time there are, to the knowledge of the author (which is fairly poor), no phenomenologically relevant interacting quantum fields known.

However, things are not that bad as they look like. Instead of searching for an explicit construction of a quantum field, which would also describe such complicated objects as bound states, one can restrict oneself to scattering phenomena which are mainly all the experiments one makes with elementary particles.

2.2 The Haag-Ruelle scattering theory

In quantum mechanics one can use the so-called Schroedinger- and Heisenberg-representation as well as the interacting- or Dirac- picture which we briefly expose: Suppose one has a Hilbert-space $\mathcal{H}$, a hamiltonian $H = H_0 + V$, where V is a perturbation of the "free" hamiltonian H_0, which is time-independent, and observables, or self-adjoint operators, A_i for some index i. In the Schroedinger-picture the states carry the time evolution: $i\frac{d}{dt}\Psi(t) = H\Psi(t)$. There exists a unitary evolution operator $U(t_2, t_1)$ which satisfies the relations $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$ and $\Psi(t_2) = U(t_2, t_1)\psi(t_2)$.
In the Heisenberg picture the states $\Psi \in \mathcal{H}$ are time-independent and the observables "carry" the time evolution $A_i^H(t) = U(t, t_1)A_i^H(t_1)U(t_1, t)$, $i\frac{d}{dt}A_i(t) = [H, A_i(t)]$.
To obtain the interacting picture one "splits" the time evolution as follows: one defines $U^I(t, t_1) \equiv e^{i(t-t_1)H_0}U(t, t_1)$ and $\Psi^I(t) \equiv U^I(t, t_1)\Psi(t_1)$. From these definitions one reeds off the time evolution of $\Psi^I(t)$:

$$i\frac{d}{dt}\Psi^I(t) = V^I(t)\Psi^I(t), \tag{2.2.1}$$

where we have written $V^I(t)$ for $e^{i(t-t_1)H_0}V(t)e^{-i(t-t_1)H_0}$. This equation gives then the general solution for $U^I(t, t_1)$:

$$U^I(t, t_0) = \sum_{j=0}^{\infty}(-i)^j \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t_j-1} dt_j V^I(t_1)\ldots V^I(t_j). \tag{2.2.2}$$

The potential is called "short-range" if the wave operator $s - lim_{t\to\infty}U(t, o)e^{-iH_0t}$

exists as a strong limit. In this case we define the scattering operator S as $S = lim_{\substack{s\to\infty \\ t\to-\infty}} e^{iH_0 t}U(t,s)e^{-iH_0 s} = lim_{\substack{s\to\infty \\ t\to-\infty}} U^I(t,s)$. This operator is called scattering operator because it relates the states at very early times (incoming states) to states at very large times (outcoming states). Both, incoming and outcoming states are supposed to be free, giving therefore the physical image of asympotically free particles colliding.

In general, in the interacting picture, the observables $A_i(t)$ are transformed to $A^I(t) \equiv e^{i(t-t_1)H_0}A_i(t)e^{-i(t-t_1)H_0}$. The time evolution for an observable in the interacting picture is then

$$i\frac{d}{dt}A_i^I(t) = [A_i^I(t), H_o] + ie^{i(t-t_1)H_0}(\partial_t A_i(t))e^{-i(t-t_1)H_0}, \tag{2.2.3}$$

which is just the free evolution in the Heisenberg picture.

It is tempting to use the interacting picture in Quantum field theory, since we know how to build free fields. Formally one can then compute 2.2.2 to the desired order and solve therefore the scattering process. But this is just not possible as indicated by the Haag no-go theorem. In the case of scalar hermitian quantum fields this theorem states:

Theorem 2.2.1. *Let there be two scalar hermitian quantum fields $\varphi_i(x)$, $i = 1, 2$, satisfying the Wightman axioms. In particular there are two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as well as two unitary representations of the orthochronous proper Poincaré group $U_1(\Lambda, a)$ and $U_2(\Lambda, a)$. Let there also be a unitary operator $V(t) : \mathcal{H}^1 \to \mathcal{H}^2$, such that $V(t)\varphi_1(x)V^{-1}(t) = \varphi_2(x)$, $V(t)U_1(\Lambda, a)V^{-1}(t) = U_2(\Lambda, a)$ and $V(t)\Omega_1 = \Omega_2$. Then if one of the fields is a free field then so is the other.*

Here the argument $t = x^0$. A proof of this theorem can be found in [27]. Similar theorems exist for spinor or vector quantum fields. This theorem then simply states

that the interacting picture only exists in the case of a free theory. In general quantum fields are only expressible in the Heisenberg picture-even the Schroedinger picture does not always exist!
However, Haag [15], Ruelle, Hepp and Araki , [16], [3] found a way to simulate the interacting picture. we will outline this theory, called the Haag-Ruelle scattering theory, in the case of a scalar hermitian self-interacting field. Proofs of all the following statement can be found in the cited articles or in [17].
Let's consider a field $\varphi(x)$ satisfying the equation of movement $(\Box + m^2)\varphi(x) = P(\varphi(x))$, where $P(.)$ is some polynomial functional in the field $\varphi(x)$ of order at least 2. We shall not discuss the problems involved by multiplying two fields at the same space-time point and will simply assume that there exists some well-defined procedure. The important thing is only that $\varphi(x)$ satisfies the Wightman axioms. However we will need a somewhat stronger form of axiom 6:

Axiom 7 (strong version of axiom 6)**.** *a)The spectrum of the operator* $P^\mu P_\mu$ *consists of two eisolated Eigen-values* $p^\mu p_\mu = 0$ *and* $p^\mu p_\mu = m^2$*, corresponding to the Eigen-spaces of the vacuum* Ω *and one particle states respectively, and a continuum* $4m^2 \leq p^\mu p_\mu \leq \infty$.
b)The three Eigen-spaces must be Poincaré invariant.
c)Let $P_{\mathcal{H}_1}$ *be the projector one the one-particle subspace. Then* $(\Omega, \varphi(x)P_{\mathcal{H}_1}\varphi(y)\Omega) = -2\pi i D_m^+(x-y)$*, where* $D_m^+(x-y) = \frac{1}{(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(p^0)e^{-i\langle p,x-y\rangle}$ *is the Jordan-Pauli function with only positive frequencies.*

Condition a) and b) is the physical requirement that there exists something as free massive particles which look as particle for any inertial observer.
c) Tells us that the fields have non-vanishing matrix elements between the physical

vacuum and $\mathcal{H}_1$.

With the help of these conditions it is possible to work through the following construction.

Consider the Fourier transform of the field $\tilde{\varphi}(p)$. Then the smeared out fields $\int d^4pf(p)\tilde{\varphi}(p)$ are bona fide operators in $\mathcal{H}$ for $f \in S(\mathbb{R}^4)$. If we restrict the support of f "around" the positive mass shell, $suppf \subset G \equiv \{p \in \mathbb{R}|0 < p^0 < \sqrt{\vec{p}^2 + 4m^2}\}$, then by axiom 7 this operator will produce out of the vacuum a one-particle-state in the Heisenberg representation. One then considers the operators

$$\varphi(g,t) \equiv \int d^4pg(p)\tilde{\varphi}(p)e^{i(\omega(p)-p^0)t}. \tag{2.2.4}$$

Now, apllied to the vacuum, these operators produce a state with "wave-function" $\tilde{\varphi}(p)e^{i(\omega(p)-p^0)t}$. Here the factor $e^{i(\omega(p)-p^0)t}$ plays the role of a "simulator" of $e^{it(H_0-H)}$ and the support properties of $\tilde{\varphi}(p)$ allow the physical one-particle state interpretation as well as the mathematical well-definedness of 2.2.4.

Intuitively one would expect that these operators converge to a sort of free states for $t \to \infty$ or $t \to -\infty$ and actually one can prove the following

Theorem 2.2.2. *The states*

$$\prod_{i=1}^{n} \varphi(g_i,t)\Omega$$

converge strongly to some states

$$u^{in}(g_1,\ldots,g_n)$$

$$(\ \textit{resp.}\ u^{out}(g_1,\ldots,g_n))$$

when $t \to -\infty$ (resp $t \to \infty$) and $g_i \in S(G)$.

The so defined states $u^{ex}(g_1,\ldots,g_n)$ do not depend on the coordinate system chosen (Poincaré-invariance).

If one of the g_i is not in $S(G)$ but in $S(\mathbb{R}^4)$, then the convergence is a weak convergence.

These asymptotic vectors span two Hilbert-spaces $\mathcal{H}^{\text{ex}}$ where "ex" stands for "in" or "out". But there is even some better result:

Theorem 2.2.3. *The linear extension of the equation*

$$a^{ex}(g)^\dagger u^{ex}(g_1, \ldots, g_n) \equiv u^{ex}(g, g_1, \ldots, g_n), \tag{2.2.5}$$

defines at most two free scalar hermitian fields $\varphi^{ex}(x)$, where

$$\varphi^{ex}(\mathcal{F}^{-1}g) \equiv a^{ex}(\bar{g}) + a^{ex}(g)^\dagger,$$

with $\mathcal{F}^{-1}g$ is the Fourier inverse transform of g (compare with equation 1.3.8). These fields transform under the same representation of the Poincaré group as $\varphi(x)$.

The fields $\varphi^{\text{ex}}(x)$ are called "asymptotic" fields since there action is only defined on $\mathcal{H}^{\text{ex}}$.
The definition of the asymptotic fields also show that they produce an irreducible operator algebra in $\mathcal{H}^{\text{ex}}$.
At this point one has to introduce the "scattering assumption":

$$\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}. \tag{2.2.6}$$

The fields $\varphi^{\text{in}}(x)$ and $\varphi^{\text{out}}(x)$ act now on the same Hilbert space and therefore there must exist a unitary operator S, called "S-matrix", Scattering-matrix or "Streuoperator", such that

$$\varphi^{\text{out}}(x) = S^{-1}\varphi^{\text{in}}(x)S. \tag{2.2.7}$$

Since the asymptotic fields are irreducibly acting on $\mathcal{H}^{\text{ex}}$ the S-matrix must be a functional of the free asymptotic fields. Its construction is the topic of the next subsection.

Remark 2.2.1. Here we have only outlined the Haag-Ruelle scattering theory for a scalar hermitian self-interacting field. But Ruelle has actually treated the general case of countably many irreducible spinor and tensor fields [21].
The "morality" of all this is that one can circumvent the no-go-theorem of Haag and the scattering matrix is in principle expressible in terms of free fields only, which is good news! This is also the reason why we are only interested in free fields.

2.3 Perturbative Construction of The S-matrix

There are several methods to construct the S-matrix. We are going to use the one of Epstein and Glaser [8]. The first steps toward this theory were done by Stueckelberg [28] and later by Bogoliubov and Shirkov [4]. We will give an outline of this method. Details can also be found in [22].
So let there be a quantum field theory invoving several fields $\varphi_i^{n_i}(x)$, where i indexes several fields and n_i is an index corresponding to the transformation properties of the ith field. Inspired by the quantum mechanical case 2.2.2 we impose the following

Scattering Assumption 1. *the S-matrix has the following perturbative expansion:*

$$S(g) = \mathbb{1} + \sum_{j=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_j T_j(x_1, \dots, x_j) g(x_1) \dots g(x_j), \tag{2.3.1}$$

where $T_j(x_1, \dots, x_j)$ is some well defined polynomial in the asymptotic fields $\varphi_i^{n_i\,ex}(x)$ and $g(x_l) \in \mathcal{S}(\mathbb{R}^4)$.

Remark 2.3.1. a) Since the T_i in 2.3.1 are polynomials in the fields $\varphi_i^{n_i \mathrm{ex}}(x)$ one expects that the S-matrix is a "matrix"-valued distribution. Therefore one needs the smearing functions $g(x_j)$.
b) It is not clear if the sum converges. Actually there are good arguments to believe that it does not! One should actually interpret the formula 2.3.1 as a formal power series. The excellent agreement with the experiment of this approach, for instance QED or the Weinberg-Salam model, justifies in its own this construction.

Since the asymptotic fields have causal (anti)commutation relations and have covariant Poincaré-transformation properties one expects the same for the S-matrix, leading us to the

Scattering Assumption 2. *a) Let* $(\Lambda, a) \in L_+^\uparrow \rtimes \mathbb{R}^4$ *Then* $U(\Lambda, a)S(g)U^{-1}(\Lambda, a) = S(g_{(\Lambda,a)})$, *where* $g_{(\Lambda,a)}(x) = g(\Lambda^{-1}(x-a))$.
b)Suppose that the test functions g_1 *and* g_2 *have disjoint support in time in some reference frame:* $suppg_1 \subset \{x \in \mathbb{M} | x^0 \in (-\infty, r)\}$, $suppg_2 \subset \{x \in \mathbb{M} | x^0 \in (r, +\infty)\}$. *Then* $S(g_1 + g_2) = S(g_1) + S(g_2)$.

It is perhaps surprising that when T_1 is given one can compute all the T_j inductively! Suppose all $T_m(x_1, \ldots, x_m);\ 1 \leq m \leq n-1$ are given and have all the desired properties. Then one first computes the quantities

$$\tilde{T}_l(X) \equiv \sum_{r=1}^{l} (-)^r \sum_{Pr} T_{n_1}(X_1) \ldots T_{n_r}(X_r),\ 1 \leq l \leq n, \tag{2.3.2}$$

where $X \equiv X_1 \cup \ldots \cup X_r$, $X_j \neq \emptyset$, $|X_i| = n_i$ and the second sum runs over all partitions of X. Then one computes the following distributions:

$$\begin{aligned} A'_n(x_1, \ldots, x_n) &= \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \\ R'_n(x_1, \ldots, x_n) &= \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \end{aligned} \tag{2.3.3}$$

where now the sum runs over all partitions $P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y$, $X \neq \emptyset$, into disjoint subsets with $|X| = n_1 \geq 1$, $|Y| \leq n-2$. We also introduce

$$D_n(x_1, \ldots, x_n) = R'_n(x_1, \ldots, x_n) - A'_n(x_1, \ldots, x_n) \tag{2.3.4}$$

If the sum in 2.3.3 ran over all partitions including the empty set then we'd had the distributions

$$\begin{aligned} A_n(x_1, \ldots, x_n) =& \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \\ =& A'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n), \\ R_n(x_1, \ldots, x_n) =& \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \\ =& R'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n), \end{aligned} \tag{2.3.5}$$

where the unknown $T_n(x_1, \ldots, x_n)$ appears. But the difference D_n is known and what remains to be done is to determine R_n or A_n.

it can be shown [22] that the distribution D_n has causal support. Moreover R_n (resp. A_n) has retarded (advanced) causal support. Therefore if one splits 2.3.4 into retarded and advanced supports one gets:

$$T_n(x_1, \ldots, x_n) = R_n(x_1, \ldots, x_n) - R'_n(x_1, \ldots, x_n). \tag{2.3.6}$$

This construction can be regarded as a well-defined time-ordered product:

$$T_n(x_1, \ldots, x_n) = T\{T_1(x_1) \ldots T_1(x_n)\}. \tag{2.3.7}$$

Here $T\{\ldots\}$ means that the product in the brackets is taken in chronological order. If all the x_i^0 are different this formula can be taken literally, if not, one has to split the distributions properly, which is the tricky part in this construction of distributions.

We must now discuss the splitting of distribution. Let $d_n^k(x) = r_n(x) - a_n(x)$ be a numerical distribution with causal support, i.e. $suppr_n \subseteq \Gamma^+_{n-1}(x_n)$ resp. $suppa_n \subseteq \Gamma^-_{n-1}(x_n)$ and $x = \{x_1, \ldots, x_n\}$. The easiest thing to do to split a distribution would be to multiply it by the step-function: $r_n(x) = \chi_n(x) d^k_n(x)$ with $\chi_n(x) = \prod_{j=1}^{n-1} \Theta(x^0_j - x^0_n)$. But this is not always allowed: Distributions do not form an algebra and one can not multiply them in impunity. We must therefore analyze the analytic properties of distribution at the splitting point. We fix this point at $x = 0$. We are allowed to do so because our distributions are translationally invariant by the second scattering assumption which ensures that one can put $x_n = 0$ and define

$$d(x) \equiv d^k_n(x_1, \ldots, x_{n-1}, 0) \in \mathcal{S}^\sharp(\mathbb{R}^m), \;\; m = 4n - 4. \tag{2.3.8}$$

The splitting point for this distribution is then $x = 0$. We introduce the useful

Definition 2.3.1. The distribution $d(x) \in \mathcal{S}^\sharp(\mathbb{R}^m)$ has a quasi-asymptotics $d_0(x)$ at $x = 0$ with respect to a positive continuous function $\rho(\delta)$, $\delta > 0$, if the limit

$$lim_{\delta \to 0} \rho(\delta) \delta^m d(\delta x) = d_0(x) \neq 0 \tag{2.3.9}$$

exists in the sense of distributions.
$d(x)$ is called singular of order ω if $lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega$.

If one can write $d(x) = d_1(x) + d_2(x)$ with $suppd_2(x)$ bounded way from 0, then $lim_{\delta \to 0} \rho(\delta) \delta^m d_1(\delta x) = d_0(x)$. The behavior of $\rho(\delta)$ will be of great importance in what follows. It can be shown that if a distribution satisfies the first part of the preceding definition then the function $\rho(\delta)$ has always a polynomial shape near 0 and the singular order is well defined. A straightforward computation shows that d_0 and d have the same singular order.
We are now ready to discuss the splitting procedure and have to distinguish two cases:
(a): $\omega < 0$: In this case we have:

$$lim_{\delta\to 0}\rho(\delta) = \infty. \tag{2.3.10}$$

Then one can show that the limit

$$lim_{\delta\to 0}\chi_0(\frac{v\dot{x}}{\delta})d(x) \equiv \Theta(v\dot{x})d(x) = r(x) \tag{2.3.11}$$

exists, where $\chi_0(t)$ is a C^∞-function over $\mathbb{R}$ growing continuously from 0 for $t = 0$ to 1 for $t \geq 1$, $v = (v_1, \ldots, v_{n-1}) \in \Gamma^+$, which means that all vectors v_j are positively time-like. The equation

$$vx = \sum_{j=1}^{n-1} v_j x_j = 0 \tag{2.3.12}$$

defines therefor a space-like hypersurface separating the causal support of d(x). This splitting is independent of the various choices of $\chi_0(t)$ and v_j.

(b): $\omega \geq 0$: In that case the limit in 3.2.3 exists only for test functions g with $D^a g(0) = 0$ for $\omega \geq |a|$. If one wants to recover a tempered distribution one has to introduce an auxiliary function $w(x) \in \mathcal{S}(\mathbb{R}^m)$ with $w(0) = 1$ and $D^a w(0) = 0$ for $|a| \leq \omega$ and define

$$r(x) = \Theta(vx)d(x)W, \tag{2.3.13}$$

where

$$Wg(x) = g(x) - w(x)\sum_{|a|=0}^{\omega} \frac{x^a}{a!}(D^a g)(0). \tag{2.3.14}$$

The so-defined distribution $r(x)$ agrees with $d(x)$ on $\Gamma^+/\{0\}$, but in contrast to the case (a) this solution is not unique and depends on the function $w(x)$ chosen. If $\tilde{r}$ is another solution constructed with another function $\tilde{w}$ then we have

$$r(x) - \tilde{r}(x) = \sum_{|a|=0}^{\omega} C_a D^a \delta^m(x). \tag{2.3.15}$$

The "splitting freedom" one has in the form of the constants C_a are not determined by causality and require other physical restrictions.

2.4 Perturbative Gauge Invariance

There is one problem in the scattering formalism we didn't discuss yet: As we have seen in chapter on, free vector fields have an unphysical sector in the Hilbert space they span. Therefore The spaces $\mathcal{H}^{\mathrm{ex}}$ also have an unphysical sector. We were also able to express the physical subspaces in term of the nilpotent operator Q (1.3.129). But the S-matrix is only required to be unitary on the physical part. It should also not send an unphysical state to a physical one and vice-versa. This is achieved by the condition

Scattering Assumption 3. *The S-matrix should satisfy*

$$[S, Q] \equiv d_Q S = 0. \tag{2.4.1}$$

This we will call "gauge-invariance" of the S-matrix.

Indeed, suppose S satisfies 3.3.3 and let Φ be a physical state. We have $\Psi \in Ker(Q) \ominus \overline{Ran(Q)}$. The equation

$$QS\Psi = SQ\Psi = 0 \tag{2.4.2}$$

implies $S\Psi \in Ker(Q)$. But $\Psi \in kerQ^\dagger$ we have

$$QS\Psi = SQ\Psi = 0 \tag{2.4.3}$$

. Since the series 2.3.1 is a formal series we demand gauge-invariance to all orders of perturbation. In first order this means:

$$[Q, T_1] = i\partial_\mu T^\mu_{1/1} = 0. \tag{2.4.4}$$

Indeed, if the commutator with the gauge charge is a divergence then by the adiabatic limit $S = lim_{g\to 1} S(g)$ this divergence term is a surface term which must vanish at ∞

and 3.3.3 holds in first order.

Taking equation 2.3.7 one formally gets for the nth order:

$$\begin{aligned} d_Q T_n =& d_Q T\{T_1(x_1)\dots T_1(x_n)\} \\ =& \sum_{l=1}^{n} T\{T_1(x_1)\dots d_Q T_1(x_l)\dots T_1(x_n)\} \\ =& \sum_{l=1}^{n} T\{T_1(x_1)\dots i\partial_\mu T^{\mu}_{1/1}(x_l)\dots T_1(x_n)\} \\ \equiv& i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^{\mu}} T^{\mu}_{n/l}(x_1,\dots,x_n). \end{aligned} \tag{2.4.5}$$

Again one should use the proper construction with the causal splitting as soon as two x_i^0 coincide and by doing this there may appear local terms $\delta(x)$. If one can absorb these terms by a suitable choice for the constants in 3.3.2 then the theory is said to be gauge invariant to the nth order.

The perturbative gauge invariance restrains considerably the choice for T_1. In the case of QED and the electro-weak theory it works beautifully.

The morality of this chapter is that:

(a) Concerning the scattering process one can work with free fields only.

(b) One needs a gauge operator Q to determine the physical subspace of the theory and to construct the S-matrix.

Chapter 3

Symmetries of the S-matrix and the Coleman-Mandula theorem

3.1 Introduction

What do we call a symmetry of a system? In general a symmetry is given by an action $\mathcal{A}$ of a group $\mathcal{G}$, which might be discrete or continuous, and which leaves the physical quantities "invariant".

By that we mean the conservation of the transition probabilities from a state Ψ to a state Φ: $(\Psi, \Phi) = (\mathcal{A}(g)\Psi, \mathcal{A}(g)\Phi)$ $\forall g \in \mathcal{G}$. A very well known theorem of E. Wigner establishes the existence of a unitary or anti-unitary representation $U(g)$ of the action of the group $\mathcal{G}$. In the case of a Lie group this representation is unitary. In the following we will use the following

Definition 3.1.1. A symmetry-group of a quantum field theory is a group $\mathcal{G}$ which has a unitary representation $U(g)$ on $\mathcal{H}$. The action of the group on the states and the operator algebra are:

$$\begin{aligned} \Psi &\mapsto U(g)\Psi, \\ \varphi(f) &\mapsto U(g)\varphi(f_g)U^{-1}(g), \end{aligned} \tag{3.1.1}$$

where $f \mapsto f_g$ is the action of $\mathcal{G}$ on the space of test functions.

We already know such a symmetry group: the group of Poincaré transformations.

3.2 Physical Symmetries

It has to be noted that not all symmetries in the sense of 3.1.1 are of physical importance. Indeed if $g \in \mathcal{G}$ and $\varphi(x,g) \equiv U(g)\varphi(x)U^{-1}(g)$ one can build asymptotic fields $\varphi^{\mathrm{ex}}(x,g)$. In general however one has then another S-matrix S_g such that

$$\varphi^{\mathrm{in}}(x,g) = S_g \varphi^{\mathrm{out}}(x,g) S_g^{-1}, \tag{3.2.1}$$

where $S_g = U(g)SU(g)^{-1}$. Taking a pragmatic point of view one would like to have only one S-matrix. We then demand that a physical symmetry group $\mathcal{G}$ satisfies:

$$U(g)S = SU(g), \ \ \forall g \in \mathcal{G}. \tag{3.2.2}$$

The question is now what conditions one must impose to obtain 3.2.2. Let us therefore introduce the definition of a Borchers class:

Definition 3.2.1. The fields $\varphi_i(x)$, i indexing the fields, belong to the same Borchers class as the field $\varphi(x)$ if they are local and relatively local.

Relative locality means that all the fields (anti)commute with one another as soon as their arguments are taken space-like. For such classes of fields one can show the

Theorem 3.2.1. *Let there be a Borcher class of fields $\varphi_j(x)$. Suppose that all the asymptotic fields $\varphi_j^{ex}(x)$ exist and form an irreducible field algebra. Then if $\varphi(x)$ belongs to the same Borcher class and $\varphi^{ex}(x)$ exists one has*

$$\varphi^{ex}(x) = \sum_j a_j(x, \partial_x) \varphi_j^{ex}(x). \tag{3.2.3}$$

The $\mathbb{C}$-functions $a_j(x, \partial_x)$ are the same for the ingoing and outgoing fields.

This theorem then asserts that all the fields in a Borchers class have the same S-matrix, since all the $a_j(x, \partial_x)$ commute with S. One can conclude that if the field $\varphi(x, g)$ is relatively local with respect to $\varphi(x)$ then one meats 3.2.2.
An important example of this is the Poincaré group. Indeed one has

$$U(a, \Lambda)\varphi(x)U^1(a, \Lambda) = \varphi(\Lambda^{-1}(x - a)), \tag{3.2.4}$$

which is relatively local with respect to $\varphi(x)$ since they obey the causal commutation relations. Therefore the generators of the Poincaré group commute with the S-matrix:

$$[P^\mu, S] = 0 = [S^{\mu\nu}, S]. \tag{3.2.5}$$

This assures us that the energy-impulsion as well as the angular moment are conserved during the scattering process.
Another important class of symmetries are the internal symmetries:

Definition 3.2.2. A symmetry group is called internal if for any compact support $\mathcal{O}$ it maps the field algebra on this compact support into the field algebra with the same support.

Otherwise stated: Internal symmetries do not "touch" the arguments of the fields. We will give another definition of internal symmetries which will be more useful:

Definition 3.2.3. An internal symmetry group is a group whose action on the Hilbert space commutes with the Poincaré action.

Since the arguments of the fields remain unchanged we have that $\varphi(x, g)$ are relatively local with respect to $\varphi(x)$ and therefore leave the S-matrix unchanged.

3.3 Charges and Conserved Currents

A special attention is given to symmetries represented by a Lie group whose generators are in turn induced by a conserved current $j^\mu_a(x)$. Here the index μ stands as usual for a for-vector whereas a stands for any other kind of possible tensorial character. The belief that only such symmetries are of physical importance comes from the old classical field theory. There one has the well-known Noether theorem which gives a correspondence between symmetry groups of the Lagrangian and conserved four-currents $j^\mu_a(x)$. These currents, when integrated, give the generators $\mathfrak{g}$ of the symmetry group in question:

$$\mathfrak{g} = \int_{x^0=0} d^3x j^0_a(x). \tag{3.3.1}$$

Because the current is conserved the integration in 3.3.1 does not depend on the time of integration. The Lorentz invariance is also maintained since equation 3.3.1 can also be written

$$\mathfrak{g} = \int_s d\sigma^\mu(x) j_{\mu a}(x), \tag{3.3.2}$$

where now s is a three dimensional space-like hyper-surface and $d\sigma^\mu(x)$ its perpendicular volume surface.

This works beautifully in classical field theory, but in quantum field theory even if we take for granted that $j^\mu_a(x)$ exists and is well defined the integration in 3.3.1 is not defined since we do not smear out the four-current which has to be a operator-valued distribution. Generally this integral diverges. However there is a way to circumvent these difficulties in some cases. We have to introduce the concept of local and quasi-local states:

Definition 3.3.1. Let $\varphi_i(x)$ $i = 1, \dots n$ be n local and relatively local quantum fields transforming under the same Poincaré representation. The states $\Psi_L \stackrel{\text{def}}{=} A_L\Omega$, with

$$A_L \stackrel{\text{def}}{=} \sum_{j=0}^{M} \int dx^1 \dots dx^j g_j(x^1, \dots, x^j)\varphi_{i^1}(x^1) \dots \varphi_{i^j}(x^j), \tag{3.3.3}$$

are called quasilocal (resp. local) if $g_j \in \mathcal{S}(x^{4j})$ (resp. $g_j \in \mathcal{D}(x^{4j})$).

In this definition $j = 0$ corresponds to a multiple of unity and $dx^l = dx_0^l dx_1^l dx_2^l dx_3^l$. A somewhat surprising result, due to the Reeh-Schlieder theorem, is that the set of local states associated with any compact region of $\mathbb{R}^4$ having a nonvanishing Lebesgue measure form a dense set in $\mathcal{H}$.

The strategy now is to give to the right handside of 3.3.2 the meaning of a sesquilinear form on a dense set of states (the local states) and investigate then if this form comes from the operator $\mathfrak{g}$.

To begin with let us define the following charge operator:

$$Q_{RT} \stackrel{\text{def}}{=} \int dx j_0(x) f_R(\vec{x}) f_T(x_0) = j_0(f_R f_T). \tag{3.3.4}$$

Here f_T and f_R are compactly supported Schwartz functions:

$$f_T(x_0) = \begin{cases} \int dx_0 f_T(x_0) = & 1, \\ f_T(x_0) = & 0 \quad \text{for } |x_0 - t| > T, \\ f_T(t + x_0) = & f_T(t - x_0) \geq 0, \end{cases}$$

and

$$f_R(\vec{x}) = \begin{cases} 1 & \text{for } |\vec{x}| < R, \\ f_R(|\vec{x}|) & \text{for } R \leq |\vec{x}| \leq R + d, \\ 0 & \text{for } |\vec{x}| \geq R + d. \end{cases}$$

The exact form of f_T and f_R is irrelevant here. The only requirements are that $f_R \to 1$ for $R \to \infty$ and $f_T \to \delta(t - x_0)$ for $T \to 0$. The naive limit $lim_{R\to\infty, T\to 0} Q_{RT}$ does not make any sense. However, we have a first positive result [7]:

Theorem 3.3.1. *Let $j^{\mu}(x)$ be a conserved quantum current, local and relatively local with respect to the quantum fields generating $\mathcal{H}$ irreducibly. Let $\mathcal{D}$ be the set of dense vectors on $\mathcal{H}$ all belonging to the domain of definition of $j^{\mu}(f)$, with $f \in \mathcal{S}$: $j^{\mu}(f)\mathcal{D} \subset \mathcal{D}$. Furthermore, let's have $(\Omega, j^{\mu}(x)\Omega) = 0$.*
Then for all localized operators A_L the commutator

$$C(A_L) \stackrel{def}{=} [Q_{RT}, A_L] \tag{3.3.5}$$

exists on $\mathcal{D}$, is a localized operator and is independent of the choice of the functions f_R and f_T for sufficiently large R and small T.

A question which arises now is if $C(A_L) = [\mathfrak{g}, A_L]$, and if one can have $Q_{RT} \to \mathfrak{g}$ in some precise way.
since $\mathfrak{g}$ is a generator of a symmetry which should leave the vacuum invariant we must have $\mathfrak{g}\Omega = 0$, as well as $(\Psi, \mathfrak{g}\Omega) = 0$, $\forall \Psi \in \mathcal{H}$. The following theorems go in this direction [7] [23]:

Theorem 3.3.2. *Under the hypothesis of theorem 3.2.1 and the additional mass-gap hypothesis the limit*

$$lim_{R\to\infty, T\to 0}(\Psi_L, Q_{RT}\Phi_L) \stackrel{def}{=} Q(\Psi_L, \Phi_L), \tag{3.3.6}$$

where Ψ_L and Φ_L are quasilocal states, exists and defines therefore a densely defined sesquilinear form.

Theorem 3.3.3. *Under the hypothesis of theorem 3.3.2*

$$lim_{R\to\infty, T\to 0}(\Psi, Q_{RT}\Omega) \stackrel{def}{=} Q(\Psi, \Omega) = 0 \tag{3.3.7}$$

for all $\Psi \in \mathcal{D}$.

Our strategy is now to investigate if there is an operator G so that $(\Psi_L, G\Phi_L) = Q(\Psi_L, \Phi_L)$ and $G = \mathfrak{g}$. We will need the helpful

Theorem 3.3.4. *Let $S(.,.)$ be a sesquilinear form which is defined on a dense set $\mathcal{D}$. Then there is an operator R defined on $\mathcal{D}$ such that $S(\Psi, \Phi) = (\Psi, R\Phi)$, $\Psi, \Phi \in \mathcal{D}$ iff for each $\Phi \in \mathcal{D}$ there exists a constant $K(\Phi) \geq 0$ independent of Ψ such that.*

$$|S(\Psi, \Phi)| \leq K(\Phi)\|\Psi\|. \tag{3.3.8}$$

Proof. $\overline{S(\Psi, \Phi)}$ is linear in Ψ for a fixed Φ. 3.3.8 then implies that $\overline{S(., \Phi)}$ is a linear continuous functional on $\mathcal{D}$ and can therefore be extended in a unique way to the whole Hilbert space. By the theorem of Riesz there is a vector $\Xi_\Phi \in \mathcal{H}$ such that

$$\overline{S(\Psi, \Phi)} = (\Xi_\Phi, \Psi). \tag{3.3.9}$$

We then define R as

$$R\Phi = \Xi_\Phi. \tag{3.3.10}$$

This operator is clearly linear and well defined on $\mathcal{D}$.

Conversely, suppose the existence of such an operator R. By the Schwartz inequality we have

$$|S(\Psi, \Phi)| = |(\Psi, R\Phi)| \leq \|\Psi\|\|R\Phi\|, \tag{3.3.11}$$

so that we may write $K(\Phi) = \|R\Phi\|$. $\square$

The following theorem will allow us to have a first positive result [19]:

Theorem 3.3.5. *The sesquilinear form $Q(.,.)$ of theorem 3.3.2 satisfies the boundedness condition 3.3.8. In this case we also have $\mathfrak{g} = G$.*

If all but the local conservation hypothesis hold then there is no boundedness condition.

In the case where there is no mass-gap in the theory, which for instance is the case in quantum electrodynamics, we still can save the construction [19]:

Theorem 3.3.6. *A sufficiant and necessary condition for the extension of the sesquilinear form $Q(.,.)$ to an operator is the local conservation of the underlying current as well as the existence of a large R such as*

$$(\Omega, [Q_{RT}, A_L]\Omega) = 0, \tag{3.3.12}$$

where A_L is a localized operator.

It is under these conditions that we will analyze the properties of physical symmetries arising from a current.

3.4 A proof of the Coleman-Mandula No-Go Theorem

Let there be a model consisting of a finite number of quantum fields satisfying the Wightman axioms. Moreover, let there be a mass-gap in the spectrum of the Energy-momentum (a model without mass-gap could, under additional assumptions, also fit in the following discussion, but would be technically more involved), and let us suppose that the asymptotic free fields exist. Finally, let us consider a locally conserved current $j^\mu(x)$. The theorems of the previous section do guaranty the existence of a symmetry-charge $\mathfrak{g}$ which arises from this current. Moreover let us suppose that this symmetry is physical, i.e. it commutes with the S-matrix. In the following, unless stated otherwise, we will always work in this scheme.

Can one make general statements about the form of this symmetries in this general case? The answer is a clear "yes". We will need the following

Lemma 3.4.1. *Let* $\mathfrak{g}$ *be an arbitrary element of the Lie-algebra of the symmetry group* $\mathcal{G}$ *and let*

$$C_n(\mathfrak{g}) \stackrel{def}{=} [P^{(1)}, [P^{(2)}, \dots [P^{(n)}, \mathfrak{g}]]], \tag{3.4.1}$$

where $P^{(i)}$*,* $i = 1, \dots, n$ *are equal to one of the* P^{μ}*. Then there is a finite* N *such that*

$$C_n(\mathfrak{g}) = 0, \ \textit{for } n \geq N. \tag{3.4.2}$$

Proof. Let us fix a finite basis in the Lie-algebra of $\mathcal{G}$

$$\{\mathfrak{g}_a : a = 1, \dots, r\}. \tag{3.4.3}$$

Then, for each $\mathfrak{g}$ in the Lie-algebra we have

$$[\mathfrak{g}, \mathfrak{g}_a] = \sum_{b=1}^{r} D_{ab}(\mathfrak{g})\mathfrak{g}_b. \tag{3.4.4}$$

One has:

$$\begin{aligned} C_n(\mathfrak{g}_a) =& [P^{(1)}, [P^{(2)}, \dots [P^{(n)}, \mathfrak{g}_a]]] \\ =& \sum_{b=1}^{r} D_{ab}(P^{(n)})[P^{(1)}, [P^{(2)}, \dots [P^{(n-1)}, \mathfrak{g}_b]]] \\ =& \sum_{b=1}^{r} (D(P^{(n)}) \dots D(P^{(1)}))_{ab}\mathfrak{g}_b. \end{aligned} \tag{3.4.5}$$

Making use of the Lie-Cartan relations for the Poincaré group and the Jacobi identity we get

$$
\begin{aligned}
[[S_{\mu\nu}, P_\nu], \mathfrak{g}_a] + [[P_\nu, \mathfrak{g}_a], S_{\mu\nu}] + [[\mathfrak{g}_a, S_{\mu\nu}], P_\nu] =& 0 \\
i\eta_{\nu\nu}[P_\mu, \mathfrak{g}_a] + \sum_{b=1}^{r} D_{ab}(P_\nu)[\mathfrak{g}_b, S_{\mu\nu}] - \sum_{b=1}^{r} D_{ab}(S_{\mu\nu})[\mathfrak{g}_b, P_\nu] =& 0 \\
i\eta_{\nu\nu}\sum_{c=1}^{r} D(P_\mu)_{ac}\mathfrak{g}_c + \sum_{c=1}^{r}\sum_{b=1}^{r}(D_{ab}(S_{\mu\nu})D_{bc}(P_\nu) - D_{ab}(P_\nu)D_{bc}(S_{\mu\nu}))\mathfrak{g}_c =& 0 \\
[D(P_\nu), D(S_{\mu\nu})] =& i\eta_{\nu\nu}D(P_\mu).
\end{aligned}
\tag{3.4.6}
$$

Replacing $S_{\mu\nu}$ by P_μ one notices that $[D(P_\mu), D(P_\nu)] = 0$. Multiplying 3.4.6 from the left by $D(P_\mu)^{n-1}$ one obtains

$$
[D(P_\nu), D(P_\mu)^{n-1}D(S_{\mu\nu})] = i\eta_{\nu\nu}D(P_\mu)^n. \tag{3.4.7}
$$

For any finite matrices A and B one has $Tr([A, B]) = 0$ and therefore

$$
Tr(D(P_\mu)^n) = 0, \quad \forall n = 1, 2, \ldots. \tag{3.4.8}
$$

Let us show that for a d-dimensional matrix H if one has $Tr(H^n)$=0 , $\forall n \in \mathbb{N}^*$ then there exists an integer q such that $H^q = 0$.

Let

$$
P(z) = det(H - z\mathbb{1}_d) = \sum_{k=0}^{d-1} \alpha_k z^k + (-z)^d \tag{3.4.9}
$$

be the characteristic polynomial and let us define

$$
Q(z) \stackrel{\text{def}}{=} det(\mathbb{1}_d - \frac{1}{z}H) = (-z)^{-d}P(z) = 1 + (-1)^d \sum_{k=0}^{d-1} \alpha_k z^{k-d}. \tag{3.4.10}
$$

Taking the logarithm we get for sufficiently large z

$$
\begin{aligned}
logQ(z) =& log\big(det(\mathbb{1}_d - \frac{1}{z}H)\big) = Tr\big(log(\mathbb{1}_d - \frac{1}{z}H)\big) \\
=& -\sum_{k=1}^{\infty}(\frac{1}{z})^k Tr(H^k) = 0,
\end{aligned}
\tag{3.4.11}
$$

and therefore $Q(z) = 1$. 3.4.10 then implies that $\alpha_k = 0$, $k = 0, \dots, n-1$ or that

$$P(z) = (-z)^d. \tag{3.4.12}$$

But since $P(z)$ is the characteristic polynomial we have $P(H) = (-H)^d = 0$, which implies the existence of an integer $q \leq d$ such that $H^q = 0$.
Applying this result to $D(P_\nu)$ we have the existence of integers n_μ such that

$$D(P_\nu)^n = 0, \text{ if } n > n_\mu. \tag{3.4.13}$$

If we note $N/4$ the maximal n_μ then for all $n \geq N$

$$D(P^{(1)}) \dots D(P^{(n)}) = 0. \tag{3.4.14}$$

This accomplishes the proof. $\square$

As an immediate consequence we have the

Corollary 3.4.2. *If*

$$V_n(\mathfrak{g}) \stackrel{def}{=} [P^2, [P^2, \dots [P^2, \mathfrak{g}]]], \tag{3.4.15}$$

then there exists an integer N such that

$$V_n(\mathfrak{g}) = 0, \forall n > N. \tag{3.4.16}$$

Proof. We have:

$$\begin{aligned}
[P^2, \mathfrak{g}_a] &= \sum_b D_{ab}(P_\mu)(P^\mu \mathfrak{g}_b + \mathfrak{g}_b P^\mu) \\
&= \sum_b (2P^\mu D(P_\mu) - D(P^\mu)D(P_\mu))_{ab} \mathfrak{g}_b; \\
[P^2, [P^2, \mathfrak{g}_a]] &= \sum_b (2P^\mu D(P_\mu) - D(P^\mu)D(P_\mu))_{ab} [P^2, \mathfrak{g}_b] \\
&= \sum_b (2P^\mu D(P_\mu) - D(P^\mu)D(P_\mu))^2_{ab} \mathfrak{g}_b,
\end{aligned} \tag{3.4.17}$$

and so on, showing that $V_n(\mathfrak{g})$ is a sum of products of $D(P_\mu)$ of at least nth order. We conclude by Lemma 3.4.1. $\square$

We will also need

Lemma 3.4.3. *Let H be an operator on a Hilbert space such that*

$$H^n = (H^\dagger)^n \tag{3.4.18}$$

for each non-negative integer n. Then if $\Psi \in D_{H^N} \cap D_H$ and $H^n\Psi = 0$ then

$$H\Psi = 0. \tag{3.4.19}$$

Proof. For N even $H^N\Psi = 0$ implies $\|H^{\frac{N}{2}}\Psi\| = 0$ because of $H^{\frac{N}{2}} = (H^\dagger)^{\frac{N}{2}}$. If $\frac{N}{2}$ is again even one itinerates the same procedure. Otherwise consider $H^{\frac{N}{2}+1}\Psi = 0$ and itinerate till $\frac{N}{2} = 1$. $\square$

We are now enabled to prove the theorem of ORaifeartaigh:

Theorem 3.4.4. *Let $\mathcal{G}$ be a Lie-group of finite order containing the Poincaré-group as a sub-group. Suppose also that P^2 and all its powers are essentially self-adjoint operators on a dense subspace $\mathcal{D}$ of $\mathcal{H}$ and that m^2 is an isolated eigen-value for P^2. Then the generators $\mathfrak{g}_a$ of the group $\mathcal{G}$ leave the eigen-space $\mathcal{H}_{m^2}$ invariant.*

Proof. Let $\Psi \in \mathcal{H}_{m^2}$ and $\mathfrak{g}$ a general element of the Lie-algebra of $\mathcal{G}$. We have:

$$(P^2 - m^2)\mathfrak{g}\Psi = [(P^2 - m^2), \mathfrak{g}]\Psi = [P^2, \mathfrak{g}]\Psi. \tag{3.4.20}$$

By repeating this N times we get by the Corollary 3.4.2

$$(P^2 - m^2)^N\mathfrak{g}\Psi = [P^2, [P^2, \dots [P^2, \mathfrak{g}]]]\Psi = 0. \tag{3.4.21}$$

Since $P^2 - m^2$ is also essantially self-adjoint on $\mathcal{D}$ we can conclude, using 3.4.3, that $(P^2 - m^2)\mathfrak{g}\Psi = 0$, or that $\mathfrak{g}\Psi \in \mathcal{H}_{m^2}$, which is what we intended to prove. $\square$

We now present the Coleman-Mandula No-Go theorem:

Theorem 3.4.5. *Let's assume the hypothesis enumerated at the beginning of this section. Let $\mathcal{G}$ be a symmetry-group of the model containing the Poincaré-group as a subgroup. Moreover suppose that $\mathcal{G}$ is a Lie-group, that its action on the fields results into local and relatively local quantum fields (this guaranties us that the symmetry-group is physically relevant). Then any element $\mathfrak{g}$ of the Lie-algebra of $\mathcal{G}$ can be written as*

$$\mathfrak{g} = a^\mu P_\mu + b^{\mu\nu} S_{\mu\nu} + \mathfrak{b}, \tag{3.4.22}$$

where $\mathfrak{b}$ is the generator of an internal symmetry group and a^μ as well as $b^{\mu\nu}$ are constants.

Remark 3.4.1. The original article of Coleman and Mandula can be found in [5]. Their argumentation is not very convincing thoug and certainly not rigorous. A satisfying proof of the general case of a finite number of quantum fields of any spinorial or tensorial character can be found in the combined work of [10], [11], [12], [9], [1] and [2].
For the sake of clarity and simplicity we will only consider a proof for a model containing a finite number of scalar fields.

Proof. Since the symmetry group's action on the fields results in local and relatively local fields it results from theorem 3.2.1 that

$$i[\mathfrak{g}, \varphi_i^{\mathrm{ex}}(x)] = \sum_j L_{ij} \varphi_j^{\mathrm{ex}}(x), \tag{3.4.23}$$

where $L_{ij} = L_{ij}(x, \partial_x)$. Note that since $\varphi_i^{\mathrm{ex}}(x)$ are free fields and due to 3.4.4 one has:

$$[\mathfrak{g}, P^2] \varphi_i^{\mathrm{ex}}(f)\Omega = 0. \tag{3.4.24}$$

Making use of the Jacobi relations as well as of the equality $[P^2, \varphi_i^{\rm ex}(x)] = -\Box\varphi_i^{\rm ex}(x)$, one is led to

$$\sum_j \Box(L_{ij}\varphi_i^{\rm ex}(x))\Omega = \sum_j (L_{ij}\Box\varphi_i^{\rm ex}(x))\Omega. \tag{3.4.25}$$

This is equivalent to

$$\sum_j (\Box L_{ij} + 2\partial^\mu L_{ij}\partial_\mu)\varphi_i^{\rm ex}(x) = 0. \tag{3.4.26}$$

But then also

$$\begin{aligned}&\sum_j (\Box_x L_{ij} + 2\partial_x^\mu L_{ij}\partial_\mu^x)[\varphi_k^{\rm ex}(x), \varphi_j^{\rm ex}(y)] \\ =&i(\Box_x L_{ik} + 2\partial_x^\mu L_{ik}\partial_\mu^x)D_m(x-y) = 0,\end{aligned} \tag{3.4.27}$$

which implies that for any solution of the Klein-Gordon equation $f(x)$ one has

$$(\Box L_{ij} + 2\partial^\mu L_{ij}\partial_\mu)f(x) = 0. \tag{3.4.28}$$

Now we are going to show that if L_{ij} satisfies 3.4.28 then it is a polynomial in $x_\mu\partial_\nu - x_\nu\partial_\mu$ and ∂_ρ.

Of course this is true if L_{ij} does not depend on x. Suppose now that L_{ij} is n^{th} degree with respect to x, i.e. (from now on we will omit the indices unless they are absolutely necessary)

$$L = x_{\mu_1}\dots x_{\mu_n}c^{\mu_1\cdots\mu_n}(\partial) + \hat{L}(x,\partial), \tag{3.4.29}$$

where $\hat{L}(x,\partial)$ is of degree $< n$ in x and $c^{\mu_1\cdots\mu_n}$ is symmetric in its indices. It is clear that $\partial^{\nu_1\cdots\nu_{n-1}}L$ satisfies 3.4.28 too. Therefore, by acting with $\partial^{\nu_1\cdots\nu_{n-1}}$ on 3.4.28 and taking into account 3.4.29 we get

$$c^{\nu_1\cdots\nu_{n-1}\mu}\partial_\mu f = 0. \tag{3.4.30}$$

Recall that f is an arbitrary solution to the Klein-Gordon equation and that c is symmetric in its indices. It follows that

$$\tilde{c}^{\mu_1\cdots\mu_n}(ip)p_{\mu_k} = 0, \quad k = 1,\dots,n \text{ for } p^2 = m^2. \tag{3.4.31}$$

Now, any polynomial satisfying this last equation can be written in the form

$$\tilde{c}^{\mu_1\ldots\mu_n}(ip)p_{\mu_k}|_{p^2=m^2} = (\frac{i}{2})^n p_{\nu_1}\ldots p_{\nu_n}\tilde{d}^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]}(ip)|_{p^2=m^2}, \tag{3.4.32}$$

where $\tilde{d}^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]}(ip)$ is antisymmetric for each couple of indices $(\nu_j\mu_j)$ and symmetric in the exchange of two such couples. Indeed, for $n=1$ one can write

$$\tilde{d}^{[\nu\mu]} = \frac{-2i}{m^2}(p^\nu\tilde{c}^\mu(ip) - p^\mu\tilde{c}^\nu(ip)). \tag{3.4.33}$$

For $n=2$

$$\begin{aligned}\tilde{d}^{[\nu_1\mu_1][\nu_2\mu_2]} =&(\frac{-2i}{m^2})^2 p^{[\nu_1}\tilde{c}^{\mu_1][\mu_2}p^{\nu_2]}\\ =&p^{[\nu_1}p^{[\nu_2}c^{\mu_1]\mu_2]},\end{aligned} \tag{3.4.34}$$

where [..] means antysymmetrization. In general:

$$\tilde{d}^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]} = (\frac{-2i}{m^2})^n p^{[\nu_1}\ldots p^{[\nu_n}\tilde{c}^{\mu_1]\ldots\mu_n]}. \tag{3.4.35}$$

Making an inverse Fourier-transform it follows that

$$\begin{aligned}L =&\frac{1}{2^n}x_{\mu_1}\ldots x_{\mu_n}\partial_{\nu_1}\ldots\partial_{\nu_n}d^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]}(\partial) + \hat{L}(x,\partial)\\ =&\frac{1}{2^{2n}}\prod_{j=1}^{n}(x_{\mu_j}\partial_{\nu_j} - x_{\nu_j}\partial_{\mu_j})d^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]}(\partial) + \hat{L}'(x,\partial),\end{aligned} \tag{3.4.36}$$

where $\hat{L}'(x,\partial)$ is the sum of $\hat{L}(x,\partial)$ plus some terms arising from the process of commutating the x_{μ_j} and the ∂_{μ_k}. It has to be noted that this does not change the fact that the term $\hat{L}'(x,\partial)$ is at most of degree $n-1$ in x. It is an easy fact to verify that

$$\frac{1}{2^{2n}}\prod_{j=1}^{n}(x_{\mu_j}\partial_{\nu_j} - x_{\nu_j}\partial_{\mu_j})d^{[\nu_1\mu_1]\ldots[\nu_n\mu_n]}(\partial) \tag{3.4.37}$$

verifies 3.4.28. Since $[\Box,(x_{\mu_j}\partial_{\nu_j} - x_{\nu_j}\partial_{\mu_j})] = 0 = [\Box,\partial_\mu]$, $\hat{L}'(x,\partial)$ verifies it too. By induction on the degree of x we therefore arrive at the result that

$$L = L(x_\mu\partial_\nu - x_\nu\partial_\mu, \partial_\rho). \tag{3.4.38}$$

Now we will first discuss the case where the generator $\mathfrak{g}$ is translationally invariant: In this case

$$L = L(\partial_\rho). \tag{3.4.39}$$

Since we have a symmetry the representation on $\mathcal{H}$ of this Lie-group is of course unitary and $\mathfrak{g}$ is (essentially) self-adjoint. Therefore all the coefficients L_{ij} are real. Applying the Jacobi relation to the operator-valued distributions $\mathfrak{g}$, $\varphi_i^{\rm ex}(x)$ and $\varphi_i^{\rm ex}(y)$ we get

$$L_{ij}(\partial) = -L_{ji}(-\partial). \tag{3.4.40}$$

Working again in Fourier-space this implies that $\tilde{L}$ must be of the form:

$$\begin{aligned} \tilde{L}(ip) &= \sum_{n_1 n_2 n_3} (b^{n_1 n_2 n_3} + \omega_p c^{n_1 n_2 n_3}) p^{n_1} p^{n_2} p^{n_3} \\ &= \sum_{n_0=0}^{1} \sum_{n_1 n_2 n_3} a^{n_0 n_1 n_2 n_3} \omega_p^{n_0} p^{n_1} p^{n_2} p^{n_3}. \end{aligned} \tag{3.4.41}$$

This is because $L(\partial)$ acts on free fields for which $p^0 = -i\partial^0 = \omega_p$ and a polynomial in ω_p can be written as a polynomial in p_i, $i = 1, 2, 3$ times $\omega_p^{n_0}$ with $n_0 = 0$ or 1. Since L must be real and because of 3.6.5 the coefficients $b_{ij}^{n_1 n_2 n_3}$ and $c_{ij}^{n_1 n_2 n_3}$ must be real and antisymmetric in ij if $n_1 + n_2 + n_3$ is even and symmetric and imaginary otherwise.We now choose the highest value among the maximal $n_0 + n_i$, $i = 1, 2, 3$. By a proper choice of coordinates, this occurs for the highest value of $n_0 + n_1 \stackrel{\rm def}{=} N_1$. We now consider the pure boost

$$\begin{cases} \omega_p' = & cosh\alpha\omega_p + sinh\alpha p_1, \\ p_1' = & sinh\alpha\omega_p + cosh\alpha p_1 \quad p_k' = p_k, \quad k = 2, 3, \\ p_\mu' = & (\Lambda^{-1}(\alpha)p)_\mu, \end{cases}$$

as well as the operator

$$\mathfrak{g}_1(\alpha) \stackrel{\rm def}{=} \frac{1}{(cosh\alpha)^{N_1}} U(\Lambda(\alpha))\mathfrak{g}U^{-1}(\Lambda(\alpha)), \tag{3.4.42}$$

and the commutation relation

$$\begin{aligned} i[\mathfrak{g}_1(\alpha), \tilde{\varphi}_i^{\text{ex}}(p)] =& \frac{i}{cosh(\alpha)^{N_1}}[U(\Lambda(\alpha))\mathfrak{g}U^{-1}(\Lambda(\alpha)), \tilde{\varphi}_i^{\text{ex}}(p)] \\ =& \frac{i}{cosh(\alpha)^{N_1}}[U(\Lambda(\alpha))\mathfrak{g}U^{-1}(\Lambda(\alpha)), U(\Lambda(\alpha))\tilde{\varphi}_i^{\text{ex}}((\Lambda^{-1}(\alpha)p))U^{-1}(\Lambda(\alpha))] \\ =& \frac{i}{cosh(\alpha)^{N_1}}U(\Lambda(\alpha))[\mathfrak{g}, \tilde{\varphi}_i^{\text{ex}}((\Lambda^{-1}(\alpha)p))]U^{-1}(\Lambda(\alpha)) \\ =& \frac{i}{cosh(\alpha)^{N_1}}\sum_j \tilde{L}_{ij}(i(\Lambda^{-1}(\alpha)p))\tilde{\varphi}_i^{\text{ex}}(p). \end{aligned} \tag{3.4.43}$$

If we let tend α to infinity and because of the presence of $\frac{i}{cosh(\alpha)^{N_1}}$, only the terms containing the expression $p_1^{N_1}$ in 3.4.41 will survive, leading us to

$$lim_{\alpha\to\infty} i[\mathfrak{g}_1(\alpha), \tilde{\varphi}_i^{\text{ex}}(p)] = \sum_j \tilde{L}_1^{\infty}(ip)_{ij}\tilde{\varphi}_j^{\text{ex}}(p), \tag{3.4.44}$$

with

$$\tilde{L}_1^{\infty}(ip)_{ij} = \sum_{n_2 n_3}(b^{N_1 n_2 n_3} + c^{(N_1-1)n_2 n_3})(\omega_p + p_1)^{N_1} p_2^{n_2} p_3^{n_3}. \tag{3.4.45}$$

Remark 3.4.2. Notice that if a term $b^{n_1 N_1 n_3} p_1^{n_1} p^{N_1} p^{n_3}$ exists in 3.4.41, that is if the highest value of $n_0 + n_1$ equals the highest value of $n_0 + n_2$, and if for such a term $n_1 < N_1$ then it disappears is not included in the following discussion.

This commutation relation defines a new operator $\mathfrak{g}_1^{\infty}$ on $\mathcal{H}^{\text{ex}}$, since the fields $\tilde{\varphi}_i^{\text{ex}}(p)$ are irreducibly represented on $\mathcal{H}^{\text{ex}}$. Up to an irrelevant constant operator we must have

$$\mathfrak{g}_1^{\infty} = \sum_{kl}\int \frac{d^3p}{2\omega_p}(-i\tilde{L}_1^{\infty}(ip)_{ij})a_k^{\text{ex}\dagger}(p)a_l^{\text{ex}}(p), \tag{3.4.46}$$

where $a_k^{\text{ex}\dagger}(p)$ and $a_l^{\text{ex}}(p)$ are the creation (resp. annihilation) operators of the free asymptotic fields. Although we did multiply to operator-valued distributions and smear it out with a polynomial function, which is not of rapid decrease, this expression is well-defined on $\mathcal{D}^{\mathrm{ex}}$ as can be checked trough a direct computation using for $a_k^{\mathrm{ex}\dagger}(p)$ and $a_l^{\mathrm{ex}}(p)$ the defining relations 1.3.3.

Next we choose the highest value of $n_2 \stackrel{\mathrm{def}}{=} N_2$ and repeat this procedure with a boost in the p_2-direction, leading us to a new operator $\mathfrak{g}_2^\infty$ and a new $\tilde{L}_2^\infty(ip)_{ij}$. The last step consists of course in itinerating this procedure for p_3 giving us an operator $\mathfrak{g}_3^\infty$ and a relation

$$i[\mathfrak{g}_3^\infty, \tilde{\varphi}_i^{\mathrm{ex}}(p)] = \sum_j \tilde{L}_3^\infty(ip)_{ij}\tilde{\varphi}_j^{\mathrm{ex}}(p), \tag{3.4.47}$$

where now

$$\tilde{L}_3^\infty(ip)_{ij} = (b^{N_1N_2N_3} + c^{(N_1-1)N_2N_3})_{ij}(\omega_p + p_3)^{N_1+N_2+N_3}. \tag{3.4.48}$$

$\tilde{L}_3^\infty(ip)_{ij}$ is thus a product of an polynomial in $(\omega_p + p_3)$ and an p-independent anti-hermitian matrix and can therefore be diagonalized giving

$$[\mathfrak{g}_3^\infty, \tilde{\psi}_i^{\mathrm{ex}}(p)] = l_{3,i}^\infty(\omega_p + p_3)^{N_1+N_2+N_3}\tilde{\psi}_i^{\mathrm{ex}}(p), \tag{3.4.49}$$

where

$$\begin{aligned} i\delta_{ij}l_{3,i}^\infty =&(R(b^{N_1N_2N_3} + c^{(N_1-1)N_2N_3})R^{-1})_{ij}, \\ \tilde{\psi}_i^{\mathrm{ex}}(p) =& \sum_j R_{ij}^{-1}\tilde{\varphi}_j^{\mathrm{ex}}(p), \end{aligned} \tag{3.4.50}$$

with $l_{3,i}^\infty \in \mathbb{R}$.

Let us now consider the expression

$$(\tilde{\psi}_i^{\mathrm{out}}(p^{(1)})\tilde{\psi}_j^{\mathrm{out}}(p^{(2)})\Omega, \mathfrak{g}_3^\infty\tilde{\psi}_i^{\mathrm{in}}(p^{(3)})\tilde{\psi}_j^{\mathrm{in}}(p^{(4)})\Omega). \tag{3.4.51}$$

Since $\mathfrak{g}_3^\infty$ is essentially self-adjoint this is also equal to

$$(\mathfrak{g}_3^\infty\tilde{\psi}_i^{\mathrm{out}}(p^{(1)})\tilde{\psi}_j^{\mathrm{out}}(p^{(2)})\Omega, \tilde{\psi}_i^{\mathrm{in}}(p^{(3)})\tilde{\psi}_j^{\mathrm{in}}(p^{(4)})\Omega). \tag{3.4.52}$$

Commuting the operator $\mathfrak{g}_3^\infty$ to the right in 3.4.51 and to the left in 3.4.52 using 3.4.49 and the fact that $\mathfrak{g}_3^\infty\Omega = 0$ we get

$$\begin{aligned}
&(\tilde{\psi}_i^{\text{out}}(p^{(1)})\tilde{\psi}_j^{\text{out}}(p^{(2)})\Omega, l_{3,i}^\infty(\omega_p + p_3^{(3)})^{N_1+N_2+N_3}\tilde{\psi}_i^{\text{in}}(p^{(3)})\tilde{\psi}_j^{\text{in}}(p^{(4)})\Omega) \\
&+(\tilde{\psi}_i^{\text{out}}(p^{(1)})\tilde{\psi}_j^{\text{out}}(p^{(2)})\Omega, \tilde{\psi}_i^{\text{in}}(p^{(3)})l_{3,j}^\infty(\omega_p + p_3^{(4)})^{N_1+N_2+N_3}\tilde{\psi}_j^{\text{in}}(p^{(4)})\Omega) \\
&= (l_{3,i}^\infty(\omega_p + p_3^{(1)})^{N_1+N_2+N_3}\tilde{\psi}_i^{\text{out}}(p^{(1)})\tilde{\psi}_j^{\text{out}}(p^{(2)})\Omega, \tilde{\psi}_i^{\text{in}}(p^{(3)})\tilde{\psi}_j^{\text{in}}(p^{(4)})\Omega) \\
&+(\tilde{\psi}_i^{\text{out}}(p^{(1)})l_{3,j}^\infty(\omega_p + p_3^{(2)})^{N_1+N_2+N_3}\tilde{\psi}_j^{\text{out}}(p^{(2)})\Omega, \tilde{\psi}_i^{\text{in}}(p^{(3)})\tilde{\psi}_j^{\text{in}}(p^{(4)})\Omega).
\end{aligned} \tag{3.4.53}$$

This yields the equality

$$\begin{aligned}
&l_{3,i}^\infty(\omega_p + p_3^{(3)})^{N_1+N_2+N_3} + l_{3,j}^\infty(\omega_p + p_3^{(4)})^{N_1+N_2+N_3} \\
&= l_{3,i}^\infty(\omega_p + p_3^{(1)})^{N_1+N_2+N_3} + l_{3,j}^\infty(\omega_p + p_3^{(2)})^{N_1+N_2+N_3}.
\end{aligned} \tag{3.4.54}$$

We are now going to make the use of a theorem in analysis due to Wichmann [6]:

Theorem 3.4.6. *Let there be for real functions $f_\alpha(p)$, $\alpha = 1, 2, 3, 4$, from which at least one is continuous. Let all these functions be defined for $p = (p_0, p_1, p_2, p_3)$, $p^\mu p_\mu = m^2$, $p_0 > 0$. Let them obey the relation*

$$f_1(p^{(1)}) + f_2(p^{(2)}) = f_3(p^{(3)}) + f_4(p^{(4)}) \tag{3.4.55}$$

for some $p^{(k)}$ in all Lorentz frames. Furthermore let there be a linear equation

$$\sum_\alpha s_\alpha p^{(\alpha)} = 0, \tag{3.4.56}$$

for some real constants s_α. Then the functions must be of the form

$$f_\alpha(p) = c_\alpha + s_\alpha u^\mu p_\mu, \tag{3.4.57}$$

with $u^\mu \in \mathbb{R}$, $\forall \mu = 0, 1, 2, 3$ and

$$\sum_\alpha c_\alpha = 0. \tag{3.4.58}$$

If we do not want 3.4.53 to be equal to 0 we must have

$$p^{(1)} + p^{(2)} = p^{(3)} + p^{(4)}, \tag{3.4.59}$$

due to the conservation of momentum in a scattering. Applying this theorem to our situation we must therefore have

$$f_i(p) = l^{\infty}_{3,i}(\omega_p + p_3)^{N_1+N_2+N_3} = u^0\omega_p + \sum_k u^k p_k. \tag{3.4.60}$$

This means that $N_1 = 1$, $N_2 = N_3 = 0$ and that $l^{\infty}_{3,i} = l^{\infty}_3$ is the same for all indices i. Considering the remark 3.4.2 and imposing Lorentz invariance we must finally have

$$\tilde{L}_{ij}(ip) = b_{ij} + ia^{\mu}p_{\mu}\delta_{ij}, \quad b_{ij} = b^{000}_{ij} = \bar{b}_{ij} = -b_{ij}, \quad a^{\mu} \in \mathbb{R}^4. \tag{3.4.61}$$

Now, it is clear that $[P^{\mu}, \tilde{\varphi}^{\text{ex}}_j(p)] = ip^{\mu}\tilde{\varphi}^{\text{ex}}_j(p)$ and that the factor b_{ij} can only represent the action of a scalar charge $\mathfrak{b}$ generating an internal symmetry. By the irreducibility of the representation of the fields on $\mathcal{H}$ we therefore have for translationally invariant symmetry generators

$$\mathfrak{g} = \mathfrak{b} + a^{\mu}P_{\mu}. \tag{3.4.62}$$

There remains to prove the theorem for a translationnaly non-invariant generator. So let $\mathfrak{g}$ be in the Lie-algebra of $\mathcal{G}$. Suppose that $\mathfrak{g}$ is translationnaly non-invariant, i.e. $[\mathfrak{g}, P^{\mu}] \neq 0$. From theorem 3.4.1 we know that there exists an $N \in \mathbb{N}$ such that

$$[P^{\mu_1}, [P^{\mu_2}, \dots [P^{\mu_n}, \mathfrak{g}]]] = 0, \tag{3.4.63}$$

for all $n > N$. This implies that

$$[P^{\mu_1}, [P^{\mu_2}, \dots [P^{\mu_{N-1}}, \mathfrak{g}]]] \tag{3.4.64}$$

is a tranlationnaly invariant non-vanishing element of the Lie-algebra. Let us first take a look at the simplest case, i.e.

$$i[\mathfrak{g}, \varphi_i^{\mathrm{ex}}(x)] = \sum_j (d^{\mu\nu}(\partial) x_{[\nu}\partial_{\mu]} + d(\partial))_{ij} \varphi_j^{\mathrm{ex}}(x). \tag{3.4.65}$$

Let us define the new generator

$$\mathfrak{g}^\mu \stackrel{\mathrm{def}}{=} i[P^\mu, \mathfrak{g}]. \tag{3.4.66}$$

Making again use of the Jacobi identity one gets

$$i[P^\mu, [\mathfrak{g}, \varphi_i^{\mathrm{ex}}(x)]] = [\mathfrak{g}, \partial^\mu \varphi_i^{\mathrm{ex}}(x)] + [\mathfrak{g}^\mu, \varphi_i^{\mathrm{ex}}(x)], \tag{3.4.67}$$

or, using 3.4.65

$$\begin{aligned} i[\mathfrak{g}^\mu, \varphi_i^{\mathrm{ex}}(x)] =& - i[\mathfrak{g}, \partial^\mu \varphi_i^{\mathrm{ex}}(x)] + i[P^\mu, [\mathfrak{g}, \varphi_i^{\mathrm{ex}}(x)]] \\ =& - \sum_j (d^{\rho\nu}(\partial) x_{[\nu}\partial_{\rho]} + d(\partial))_{ij} \partial^\mu \varphi_j^{\mathrm{ex}}(x) \\ &+ \partial^\mu \sum_j (d^{\rho\nu}(\partial) x_{[\nu}\partial_{\rho]} + d(\partial))_{ij} \varphi_j^{\mathrm{ex}}(x) \\ =& 2 \sum_j (d^{\rho\mu}(\partial)\partial_\rho)_{ij} \varphi_j^{\mathrm{ex}}(x). \end{aligned} \tag{3.4.68}$$

But this is precisely the case we already discussed and we know that

$$\mathfrak{g}^\mu = \mathfrak{b} + a^{[\nu\mu]} P_\nu, \tag{3.4.69}$$

the antisymmetry of $a^{[\nu\mu]}$ coming from the fact that $d^{\rho\mu}(\partial) = d^{\rho\mu}$ is antisymmetric in its indices too. Equation 3.4.65, 3.4.66, 3.4.69 and the fact that

$$[S_{\mu\nu}, P_\rho] = i(\eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu), \tag{3.4.70}$$

as well as

$$i[S_{\mu\nu}, \varphi_i^{\mathrm{ex}}(x)] = (x_{[\mu}\partial_{\nu]}) \varphi_i^{\mathrm{ex}}(x), \tag{3.4.71}$$

leave as only possibility

$$\begin{aligned} \mathfrak{g} =& \mathfrak{b} + a^{\nu\mu} S_{\nu\mu} + a^{\mu} P_{\mu}, \\ a^{\nu\mu} =& - a^{\mu\nu} = \bar{a}^{\nu\mu}, \\ a^{\mu} \in& \mathbb{R}^4. \end{aligned} \tag{3.4.72}$$

The next case is L being of second degree with respect to $x_{[\mu}\partial_{\nu]}$, i.e.

$$\begin{aligned} i[\mathfrak{g}, \varphi_i^{\text{ex}}(x)] =& \sum_j \big((x_{[\mu}\partial_{\nu]})(x_{[\rho}\partial_{\lambda]}) d^{[\mu\nu][\rho\lambda]}(\partial)_{ij} \\ & + (x_{[\mu}\partial_{\nu]}) d^{\mu\nu}(\partial)_{ij} + d(\partial)_{ij}\big) \varphi_j^{\text{ex}}(x). \end{aligned} \tag{3.4.73}$$

One then considers the generator

$$\mathfrak{g}^{\mu\nu} \stackrel{\text{def}}{=} [P^{\mu}, [P^{\nu}, \mathfrak{g}]], \tag{3.4.74}$$

which is easily seen to be translationnally invariant and must obey

$$[\mathfrak{g}^{\mu\nu}, \tilde{\varphi}_i^{\text{ex}}(ip)] = - \sum_j 8 p_{\rho} p_{\lambda} \tilde{d}^{[\mu\rho][\nu\lambda]}(ip)_{ij} \tilde{\varphi}_j^{\text{ex}}(ip). \tag{3.4.75}$$

We know that the coefficient on the right hand side can at most be of first degree in p. As a consequence

$$\tilde{d}^{[\mu\rho][\nu\lambda]}(ip)_{ij} = 0 \tag{3.4.76}$$

and 3.4.73 reduces to 3.4.65, the solution of which is again 3.4.72.

The generalization to L of degree higher than 2 in $(x_{[\rho}\partial_{\lambda]})$ is straightforward.

This accomplishes the proof to the Coleman-Mandula No-Go theorem. □

3.5 Supersymmetry

The just enounced No-Go theorem tells us, that it is impossible to mix non-trivially the geometrical (or relativistic) symmetries with the internal ones. To obtain this result we implicitly assumed that space time was $\mathbb{R}^4$ and that our charges for the symmetry group where so-called bosonic charges. This means that the action of the symmetry on the various fields went through a commutator,

$$[\mathfrak{g}, \varphi(x)], \tag{3.5.1}$$

and that this action left the tensorial properties of the fields unchanged.

A way to circumgo the Coleman-Mandula No-Go-theorem is to consider also fermionic generators for a symmetry-group, which then is no longer a usual Lie-group, but a so-called super-Lie-group. We then also have a super-Lie-algebra generating this group having bosonic as well as fermionic operators.

Consider for instance a fermionic generator $\mathfrak{g}_F$, where F is a collection of spin-indexes. Then

$$\begin{aligned}[][\mathfrak{g}_F, \varphi_B(x)] =& \text{a linear combination of spinorial fields and their derivatives,} \\ [\mathfrak{g}_F, \varphi_{F'}(x)] =& \text{a linear combination of bosonic fields and their derivatives.}\end{aligned} \tag{3.5.2}$$

This means that, unlike to the previous case, the symmetry generators intertwine bosinic and fermionic fields appearing in the theory if the considered generators are of spinorial type.

Let us take the simplest case where in adition to the bosonic generators one has a couple of fermionic generators Q_a, $a = 1, 2$, and their adjoints, $\bar{Q}_{\bar{a}}$. Then we have

$$\begin{aligned}\{Q_a, \bar{Q}_{\bar{b}}\} =& \sigma^{\mu}_{a\bar{b}} c_\mu, \\ \{Q_a, Q_b\} =& d_{ab}.\end{aligned} \tag{3.5.3}$$

By virtue of the Coleman-Mandula theorem, the first equality forces us to take $c_\mu = cP_\mu$. Furthermore, taking adjoints on both sides implies that $c = \in \mathbb{R}$ and positive definiteness of the scalar product in the Hilbert space implies $c > 0$. Without loss of generality we can put $c = 2$.

Next we show that the $Q'_a s$ are translationally invariant. First we establish 3.4.1 also for a fermionic charge Q. Examine the equality

$$i[Q, \varphi_B(x)] = f(x, \partial)\varphi_F(x), \tag{3.5.4}$$

where $f(x, \partial)$ is a polynomial of order k in x. Here we omit all indices for the sake of clarity. One has

$$[P_\lambda, [Q, \varphi_B(x)]] = -if(x, \partial)[Q, \varphi_B(x)] = -f(x, \partial)\partial_\lambda \varphi_F(x). \tag{3.5.5}$$

From the Jacobi identity follows

$$f(x, \partial)\partial_\lambda \varphi_F(x) - i\partial_\lambda [Q, \varphi_B(x)] = -[Q^{(1)}_\lambda, \varphi_B(x)], \tag{3.5.6}$$

where we have written $Q^{(1)}_\lambda$ for $[P_\lambda, Q]$. By 3.5.4 we also have

$$i\partial_\lambda [Q, \varphi_B(x)] = (\partial_\lambda f)\varphi_F(x) + f\partial_\lambda \varphi_F(x). \tag{3.5.7}$$

Together with 3.5.6 this gives

$$i[Q^{(1)}_\lambda, \varphi_B(x)] = i\partial_\lambda f \varphi_F(x). \tag{3.5.8}$$

Repeating this $k - 1$ times gives

$$i[Q^{(k)}_{\lambda_1 \ldots \lambda_k}, \varphi_B(x)] = i^k \partial^k_{\lambda_1 \ldots \lambda_k} f \varphi_F(x). \tag{3.5.9}$$

Of course $\partial^k_{\lambda_1 \ldots \lambda_k} f$ no longer depends on x and therefore, there exists l such that

$$Q^{(l)}_{\lambda_1 \ldots \lambda_l} = [P_{\lambda_1}, [\ldots, [P_{\lambda_l}, Q]]] = 0. \tag{3.5.10}$$

Now we are going to show that $l = 0$. Consider

$$[P_\lambda, \{Q_a, \bar{Q}_{\bar{b}}\}] = 2[P_\lambda, P_\mu]\sigma^\mu_{a\bar{b}} = 0, \tag{3.5.11}$$

or, by the Jacobi relations,

$$\{Q^{(1)}_{a\lambda_1}, \bar{Q}_{\bar{b}}\} + \{Q_a, \bar{Q}^{(1)}_{\bar{b}\lambda_1}\} = 0. \tag{3.5.12}$$

This can straightforwardly be generalized to

$$\sum_{n=0}^{l} \binom{l}{n} \{i^{l-n} Q^{(l-n)}_{a\underbrace{\lambda_1, \ldots, \lambda_1}_{l-n}}, ((i^n Q^{(n)})\dagger)_{\bar{b}\underbrace{\lambda_1, \ldots, \lambda_1}_{n}}\} = 0, \quad l \geq 1. \tag{3.5.13}$$

Suppose now that there is a $d \geq 0$ such that $Q^{(d)}_{a\underbrace{\lambda_1, \ldots, \lambda_1}_{d}} \neq 0$ and $Q^{(d+1)}_{a\underbrace{\lambda_1, \ldots, \lambda_1}_{d+1}} = 0$. If we chose $l = 2d$ in 3.5.13 we get

$$\binom{2d}{d} \{Q^{(d)}_{a\underbrace{\lambda_1, \ldots, \lambda_1}_{d}}, (Q^{(d)}\dagger)_{\bar{b}\underbrace{\lambda_1, \ldots, \lambda_1}_{d}}\} = 0. \tag{3.5.14}$$

By the positive definiteness of the scalar product in $\mathcal{H}$ this implies that $Q^{(d)}_{a\underbrace{\lambda_1, \ldots, \lambda_1}_{d}} = 0$, which contradicts our hypothesis and therefore we must have that $[P_\mu, Q_a] = 0$. We now investigate the second relation in 3.5.3. Because of the spinorial transformation properties we have

$$\{Q_a, Q_b\} = z\epsilon_{ab} + b[\sigma^\mu, \bar{\sigma}^\nu]_{ab} M_{\mu\nu}, \tag{3.5.15}$$

where b and z are numerical factors independent of x, since the Q_a's are translationally invariant. But the $M_{\mu\nu}$ are not translationally invariant and therefore

$$\{Q_a, Q_b\} = z\epsilon_{ab}. \tag{3.5.16}$$

Now the righthand side is symmetric in the indices a and b whereas the lefthand side is antisymmetric and therefore we must have $z = 0$. The anti-commutator relation of our super-generators now read

$$\begin{aligned}\{Q_a, \bar{Q}_{\bar{b}}\} =& 2\sigma^{\mu}_{a\bar{b}} P_\mu,\\ \{Q_a, Q_b\} =& \{\bar{Q}_{\bar{a}}, \bar{Q}_{\bar{b}}\} = 0,\\ [P_\mu, Q_b] =& [P_\mu, \bar{Q}_{\bar{b}}] = 0,\\ [Q_a, M^{\mu\nu}] =& [\sigma^\mu, \bar{\sigma}^\nu]_a^b Q_b.\end{aligned} \tag{3.5.17}$$

It should be clear that this commutation relation go over mutatis mutandi to the asymtotic free Hilbert spaces $\mathcal{H}^{\text{ex}}$.

3.6 Representations of The Supersymmetrygroup

We study now the representations of the just established super-generators. We restrain ourselves to the massive case with only one pair of super-generators $Q_a, \bar{Q}_{\bar{b}}$. The Hilbert spaces carrying a representation of this group carry also a representation of the proper orthochronous Lorentz group, since the latter is contained in the supergroup.

So let $u(S,S) \in \mathcal{H}$ be the state at rest with mass m and the highest spin-value $S = S_3$:

$$\begin{aligned}P_\mu P^\mu u(S,S) =& m^2 u(S,S),\\ P^0 u(S,S) =& m u(S,S), \quad P_i u(S,S) = 0,\\ W^3 u(S,S) \overset{\text{def}}{=}& \frac{-1}{2m} \epsilon^{3\alpha\beta\gamma} P_\alpha M_{\beta\gamma} u(S,S) = S u(mS,S),\\ W^\mu W_\mu u(S,S) =& -S(S+1).\end{aligned} \tag{3.6.1}$$

Since the argumentation takes place in the rest frame, the various operators simplify to

$$\begin{aligned}W_\mu =& (0, S_1, S_2, S_3),\\ W_\mu W^\mu =& -(S_1)^2 - (S_2)^2 - (S_3)^2,\end{aligned} \tag{3.6.2}$$

where S_i are the usual spin-operators.

Now, the commutation relations of the algebra yell:

$$\begin{aligned}[S_3, Q_2]u(S,S) =&(S_3 - S)Q_2u(S,S) \\ =&-\frac{1}{2}(\sigma^3_{21}Q_1 + \sigma^3_{22}Q_2)u(S,S) \\ =&\frac{1}{2}Q_2u(S,S),\end{aligned} \tag{3.6.3}$$

and therefore

$$S_3Q_2u(s,s) = (S + \frac{1}{2})Q_2u(S,S), \tag{3.6.4}$$

which, since S is the highest spin-value, is only possible if $Q_2u(S,S) = 0$. Similarly one proves that $\bar{Q}_{\bar{1}}u(S,S) = 0$, and that

$$\begin{aligned}S_3Q_1u(S,S) =&(S - \frac{1}{2})Q_1u(S,S), \\ S_3\bar{Q}_{\bar{2}}u(S,S) =&(S - \frac{1}{2})\bar{Q}_{\bar{2}}u(S,S).\end{aligned} \tag{3.6.5}$$

Therefore the vectors $Q_1u(S,S)$ and $\bar{Q}_{\bar{2}}u(S,S)$ have total spin $(S - \frac{1}{2})$, which is also their value for S_3.

Since in the rest frame one has $\{Q_1, \bar{Q}_{\bar{2}}\} = 0$, there is one more state which is possible to reach with the Supergenerators: $Q_1\bar{Q}_{\bar{2}}u(S,S)$. It can be shown that this state has again S as total spin and $S-1$ as an Eigenvalue for S_3.

Continuing similarly with the remaining spin-states occurring in the theory one is able to build irreducible representations for the Supersymmetry-group. But we will turn back to explicit calculation in the following chapter.

Chapter 4

Construction of the superfields

4.1 Introduction

After all this introducing and known material it's time to discuss our construction of the superfields. It must be stressed that we won't use any reference to a "classical" theory. Instead we immediatly define a full quantum construction of superfields.

Firstly we are going to define the action of the super-algebra on the asymptotic fields acting on the Hilbert spaces $\mathcal{H}^{ex}$ (From now on we will drop the subscripts "^{ex}" and just write $\mathcal{H}$). We can think of the fermionic charges Q_a and $\bar{Q}_{\bar{b}}$ in our symmetry super-Lie group as generators in a super-space as the usual Poincaré generators P_μ are translation generators in the Minkowski-space $\mathcal{M}$. But the coordinates in our super-space must also be spinors θ_a and $\bar{\theta}_{\bar{b}}$, such that θQ become invariant translations as $a_\mu P^\mu$ are invariant for $a^\mu \in \mathcal{M}$. Such objects are called Grassmann numbers. They anticommute and therefore a function on a superspace can always be written as a polynomial not exceeding the fourth degree in the Grassman numbers times a function on $\mathcal{M}$.

Let therefore $\mathcal{M} \times (\theta_a, \bar{\theta}_{\bar{b}})$ be the definition of the superspace. Then a quantum superfield is a quantum field defined as an operator-valued quantum field on $\mathcal{M} \times (\theta_a, \bar{\theta}_{\bar{b}})$.

In ordinary quantum field theory defined on the Minkowsky-space M we learned that there exists unitary translation operators $U(\mathbb{1}, a)$ such that

$$\varphi(x+a) = U(\mathbb{1}, a)\varphi(x)U(\mathbb{1}, -a). \tag{4.1.1}$$

We want to construct the superfields the same way. We use the quantities $\theta^a Q_a$ and $\bar{\theta}^{\bar{b}}\bar{Q}_{\bar{b}}$ as the translation operators in superspace in the direction θ^a and $\bar{\theta}^{\bar{b}}$ respectively. Note that the adjoint of $\theta^a Q_a$ is $\bar{Q}_{\bar{a}}\bar{\theta}^{\bar{a}} = -\bar{\theta}^{\bar{b}}\bar{Q}_{\bar{b}}$, because $\bar{Q}_{\bar{a}}$ and $\bar{\theta}^{\bar{a}}$ where chosen to anticommute. If we want to construct a unitary translation in superspace one has therefore to chose the following

Definition 4.1.1. The translation operator in superspace $W(\theta, \bar{\theta})$ and the free superfields $\Phi_A(x, \theta, \bar{\theta})$ are given by
1) $W(\theta, \bar{\theta}) \overset{\text{def}}{=} exp(i\theta^a Q_a - i\bar{\theta}^{\bar{b}}\bar{Q}_{\bar{b}}) = exp(i\theta Q + i\bar{\theta}\bar{Q})$.
2) $\Phi_A(x, \theta, \bar{\theta}) \overset{\text{def}}{=} W(\theta, \bar{\theta})\varphi_A(x)W^{-1}(\theta, \bar{\theta})$.

Remark 4.1.1. The fields $\varphi_A(x)$ are ordinary free quantum fields. They may be scalar, spinorial or tensorial, depending on the index A.

We are now going to construct the various superfields we are going to use.

4.2 The Scalar Superfield

Here we are going to use the charged, or complex, Klein-Gordon field $\varphi(x)$ with mass m from which we are going to build the superfield $\Phi(x, \theta, \bar{\theta})$ in the spirit of definition

4.1.1. Note first that the (anti)commutation relations of the superalgebra imply

$$[\theta^a Q_a, \bar{\theta}^{\bar{b}} \bar{Q}_{\bar{b}}] = -2\theta\sigma^\mu\bar{\theta}P_\mu. \tag{4.2.1}$$

This commutator in turn commutes with θ^a, $\bar{\theta}^{\bar{b}}$, Q_a and $\bar{Q}_{\bar{b}}$. Therefore, using the Hausdorff formula, one can write

$$W(\theta,\bar{\theta}) = e^{i\theta Q} e^{i\bar{\theta}\bar{Q}} e^{\theta\sigma^\mu\bar{\theta}P_\mu}, \tag{4.2.2}$$

and similarly for $W^{-1}(\theta,\bar{\theta})$. The superfield now reads:

$$\Phi(x,\theta,\bar{\theta}) = e^{i\theta Q} e^{i\bar{\theta}\bar{Q}} e^{\theta\sigma^\mu\bar{\theta}P_\mu} \varphi(x) e^{-\theta\sigma^\mu\bar{\theta}P_\mu} e^{-i\bar{\theta}\bar{Q}} e^{-i\theta Q}. \tag{4.2.3}$$

We first compute the middle term $\Phi'(x,\theta,\bar{\theta}) \stackrel{\text{def}}{=} e^{\theta\sigma^\mu\bar{\theta}P_\mu}\varphi(x)e^{-\theta\sigma^\mu\bar{\theta}P_\mu}$. Using $\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = \frac{1}{2}\eta^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}$ this becomes:

$$\begin{aligned}\Phi'(x,\theta,\bar{\theta}) =& e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu}\varphi(x)\\ =&\varphi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\varphi(x)\\ =&\varphi(y),\end{aligned} \tag{4.2.4}$$

where we have defined $y \equiv x + i\theta\sigma\bar{\theta}$.

To continue this computation we need the commutators of $\varphi(x)$ with Q_a and $\bar{Q}_{\bar{b}}$. For this purpose we introduce the following

Definition 4.2.1. The covariant supersymmetric derivatives D_a and $\bar{D}_{\bar{b}}$ are defined as follows:

$D_a \equiv \frac{\partial}{\partial\theta^a} - i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu$, $\bar{D}_{\bar{a}} \equiv \frac{\partial}{\partial\bar{\theta}^{\bar{a}}} - i\bar{\sigma}^\mu_{b\bar{a}}\theta^b\partial_\mu$

Lemma 4.2.1. *The operators D_a and $\bar{D}_{\bar{b}}$ satisfy the following relations:*

$$\begin{aligned} i)\{D_a, D_b\} =& \quad 0, \{\bar{D}_{\bar{a}}, \bar{D}_{\bar{b}}\} = \quad 0, \{D_a, \bar{D}_{\bar{b}}\} = -2i\sigma^\mu_{a\bar{b}}\partial_\mu,\\ ii)D_a\Phi(x,\theta,\bar{\theta}) =& \ i[Q_a, \Phi(x,\theta,\bar{\theta})] = \ iW(\theta,\bar{\theta})[Q_a,\varphi(x)]W^{-1}(\theta,\bar{\theta}),\\ iii)\bar{D}_{\bar{a}}\Phi(x,\theta,\bar{\theta}) =& \ i[\bar{Q}_{\bar{a}}, \Phi(x,\theta,\bar{\theta})] = \ iW(\theta,\bar{\theta})[\bar{Q}_{\bar{a}},\varphi(x)]W^{-1}(\theta,\bar{\theta}).\end{aligned} \tag{4.2.5}$$

Proof. All these results follow from straightforward computations. □

In the usual literature fields satisfying $\bar{D}_{\bar{b}}\Phi(x,\theta,\bar{\theta}) = 0$ (resp. $D_a\Phi(x,\theta,\bar{\theta}) = 0$) are called (anti-)chiral superfields. So to obtain a anti-chiral superfield we must impose

$$[Q_a, \varphi(x)] = 0 \text{ and } [Q_a, \varphi^\dagger(x)] = -i\sqrt{2}\psi_a(x). \tag{4.2.6}$$

We will concentrate on the anti-chiral field, since it is the one we are going to use. Of course, one could construct the chiral one following the same procedure but by undertaking the obvious changes.

Because of the transformation properties of Q_a the field $\psi_a(x)$ must be a $(\frac{1}{2}, 0)$ two-component spinor field. The factor $-\sqrt{2}i$ is conventional.

It has to be noted that $\varphi(x)$ and $\psi_a(x)$ are relatively local, for

$$\begin{aligned}
[\varphi(x), \psi_a(y)] =& \frac{i}{\sqrt{2}}\left[\varphi(x), [Q_a, \varphi^\dagger(y)]\right] \\
=& \frac{i}{\sqrt{2}}\left[[\varphi(x), \varphi^\dagger(y)], Q_a\right] + \frac{i}{\sqrt{2}}\left[\varphi^\dagger(y), [Q_a, \varphi(x)]\right] \\
=& \frac{1}{\sqrt{2}}[D_m(x-y), Q_a] + 0 = 0,
\end{aligned} \tag{4.2.7}$$

since $D_m(x-y)$ is an $\mathbb{1}$-valued distribution which commutes with any other operator. Replacing $\varphi(x)$ by $\varphi^\dagger(x)$ one similarly finds that $[\varphi^\dagger(x), \psi_a(y)] = 0$.

The field $\psi_a(x)$ is a solution of the Klein-Gordon equation because $\varphi(x)$ is a free (asymptotic) field. As we have seen in chapter one, a spinor field satisfying the Klein-Gordon equation can be seen as a sum of two Majorana fields. Therefore one can take $\psi_a(x)$ to be a Majorana field:

$$\begin{aligned}
i\sigma^\mu_{a\bar{b}}\partial_\mu\bar{\psi}^{\bar{b}}(x) =& m\psi_a(x); \\
\{\psi_a(x), \psi_b(y)\} =& im\epsilon_{ab}D_m(x-y), \\
\{\psi_a(x), \bar{\psi}_{\bar{b}}(y)\} =& \sigma^\mu_{a\bar{b}}\partial_\mu D_m(x-y).
\end{aligned} \tag{4.2.8}$$

Next we have to investigate the anti-commutator $\{Q_a, \psi_b(x)\}$. Using the Jacobi identities one gets:

$$\begin{aligned}
\{Q_a, \psi_b(x)\} =& \frac{i}{\sqrt{2}}\{Q_a, [Q_b, \varphi^\dagger(x)]\} \\
=& \frac{i}{\sqrt{2}}\{Q_b, [\varphi^\dagger(x), Q_a]\} - \frac{i}{\sqrt{2}}\left[\varphi^\dagger(x), \{Q_a, Q_b\}\right] \\
=& -\frac{i}{\sqrt{2}}\{Q_b, [Q_a, \varphi^\dagger(x)]\} = -\{Q_b, \psi_a(x)\}.
\end{aligned} \tag{4.2.9}$$

Therefore one concludes that $\{Q_a, \psi_b(x)\}$ must be proportional to $\epsilon_{ab} f(x)$, with $f(x)$ a free scalar field. From (4.2.8), (4.2.7) and the Jacobi relations one gets

$$\begin{aligned}
[f(x), \varphi^\dagger(y)] =& \frac{-1}{2}\left[\{Q_a, \psi^a(x)\}, \varphi^\dagger(y)\right] \\
=& \frac{-1}{2}\{Q_a, [\psi^a(x), \varphi^\dagger(y)]\} + \frac{1}{2}\{\psi^a(x), [\varphi^\dagger(y), Q_a]\} \\
=& \frac{i\sqrt{2}}{2}\{\psi^a(x), \psi_a(y)\} = -\sqrt{2} m D_m(x-y),
\end{aligned} \tag{4.2.10}$$

from which we conclude that

$$f(x) = -i\sqrt{2} m \varphi(x). \tag{4.2.11}$$

for esthetic reasons we will rename $\sqrt{2}\varphi(x)$ just $\varphi(x)$ obtaining the

Definition 4.2.2. The components of the anti-chiral superfiled verify the following (anti)commutaton relations:

$$\begin{aligned}
[\varphi(x), \varphi^\dagger(y)] =& \quad -2iD_m(x-y), \quad [\varphi(x), \varphi(y)] = & 0; \\
\{\psi_a(x), \psi_b(y)\} =& \; im\epsilon_{ab} D_m(x-y), \{\psi_a(x), \bar{\psi}_{\bar{b}}(y)\} = & \sigma^\mu_{a\bar{b}} \partial_\mu D_m(x-y); \\
i[Q_a, \varphi(x)] =& \; 0; i[Q_a, \varphi^\dagger(x)] = & 2\psi_a; \\
i\{Q_a, \psi_b(x)\} =& \; m\epsilon_{ab}\varphi(x); i\{Q_a, \bar{\psi}_{\bar{b}}(x)\} = & i\sigma^\mu_{a\bar{b}}\partial_\mu \varphi(x).
\end{aligned} \tag{4.2.12}$$

Now we can continue the calculation (4.2.4) of the superfield. First we note that

$$\begin{aligned}
e^{i\bar{\theta}\bar{Q}} \varphi(x) e^{-i\bar{\theta}\bar{Q}} =& \varphi(x) + i[\bar{\theta}\bar{Q}, \varphi(x)] + \frac{1}{2}\left[\bar{\theta}\bar{Q}, [\varphi(x), \bar{\theta}\bar{Q}]\right] \\
=& \varphi(x) + 2\bar{\theta}\bar{\psi}(x) - m\bar{\theta}\bar{\theta}\varphi^\dagger(x),
\end{aligned} \tag{4.2.13}$$

and also

$$e^{i\theta Q}\varphi^\dagger(x)e^{-i\theta Q} = \varphi^\dagger(x) + 2\theta\psi(x) - m\theta\theta\varphi(x). \tag{4.2.14}$$

We also need to compute

$$e^{i\theta Q}\bar\theta\bar\psi(x)e^{-i\theta Q} = \bar\theta\bar\psi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\varphi(x). \tag{4.2.15}$$

Putting all this together we arrive at the

Result 1. *The chiral superfield $\Phi(x,\theta,\bar\theta)$ is given by:*

$$\begin{aligned}\Phi(\theta,\bar\theta,x) =&(1 + i\theta\sigma^\mu\bar\theta\partial_\mu + \frac{m^2}{4}\theta\theta\bar\theta\bar\theta)\varphi(x)\\ &+2(1 - i\theta\sigma^\mu\bar\theta\partial_\mu)\bar\theta\bar\psi(x) - 2m\bar\theta\bar\theta\theta\psi(x)\\ &-m\bar\theta\bar\theta\varphi^\dagger(x).\end{aligned} \tag{4.2.16}$$

If we start with the spinor component $\psi_a(x)$ we obtain

$$\begin{aligned}\Psi_a(x,\theta,\bar\theta) \overset{def}{=}&W(\theta,\bar\theta)\psi_a(x)W^{-1}(\theta,\bar\theta)\\ =&(1 - i\theta\sigma^\mu\bar\theta\partial_\mu + \frac{m^2}{4}\theta\theta\bar\theta\bar\theta)\psi_a(x)\\ &-m\theta_a(1 - i\theta\sigma^\mu\bar\theta\partial_\mu)\varphi(x)\\ &-i\sigma^\nu_{a\bar b}\bar\theta^{\bar b}\partial_\nu(1 - i\theta\sigma^\mu\bar\theta\partial_\mu)(\varphi^\dagger(x) + 2\theta\psi(x))\\ &+im\theta\theta\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\partial_\mu\varphi(x).\end{aligned} \tag{4.2.17}$$

Here we see that the highest spin-value is ($S = \frac{1}{2}$). By the discussion at the end of chapter 3 we conclude that we have the following structure for the irreducible representation:

$$\begin{aligned}u(\frac{1}{2},\frac{1}{2}) \xrightarrow{Q_1} u_1(0,0) \xrightarrow{\bar Q_{\bar 2}} u(\frac{1}{2},-\frac{1}{2}),\\ u(\frac{1}{2},\frac{1}{2}) \xrightarrow{\bar Q_{\bar 2}} u_2(0,0) \xrightarrow{Q_1} u(\frac{1}{2},-\frac{1}{2}).\end{aligned} \tag{4.2.18}$$

Such a representation is labelled by Ω_0.

4.3 The Chiral Ghost-Superfield

Here we proceed to the construction of a superfield starting with a scalar ghost component $u(x) \stackrel{\text{def}}{=} u_1(x)+u_2(x)$, where $u_i(x)$ are self-conjugate ghost-fields. The differences with the previous section is that now the scalar components have anti-commutation relations and must therefore be treated as fermionic fields whereas the spinor ghost-fields have commutation-relations and must be seen as bosonic components.

Definition 4.3.1. The (anti)-commutation relation of the super-algebra with the ghost field components are:

$$\begin{aligned} i\{Q_a, u(x)\} =&0, & i\{Q_a, u(x)^K\} \quad =&2i\chi_a(x), \\ i[Q_a, \chi_b(x)] =&-im\epsilon_{ab}u(x), & i[Q_a, \bar{\chi}_{\bar{b}}(x)] \quad =&\sigma^\mu_{a\bar{b}}\partial_\mu u(x). \end{aligned} \tag{4.3.1}$$

Making the same computations as before we arrive at the

Result 2. *The ghost chiral superfield is explicitly given by:*

$$\begin{aligned} U(x,\theta,\bar\theta) \stackrel{def}{=}& W(\theta,\bar\theta)u(x)W^{-1}(\theta,\bar\theta) \\ =&\big(1+i\theta\sigma^\mu\bar\theta\partial_\mu+\frac{m^2}{4}\theta\theta\bar\theta\bar\theta\big)u(x) \\ &+\big(1-i\theta\sigma^\mu\bar\theta\partial_\mu\big)2i\bar\theta\bar\chi(x)-m\bar\theta\bar\theta u^K(x) \\ &-2im\bar\theta\bar\theta\theta\chi(x), \end{aligned} \tag{4.3.2}$$

$$\begin{aligned} X_a(x,\theta,\bar\theta) \stackrel{def}{=}& W(\theta,\bar\theta)\chi_a(x)W^{-1}(\theta,\bar\theta) \\ =&\big(1-i\theta\sigma^\mu\bar\theta\partial_\mu+\frac{m^2}{4}\theta\theta\bar\theta\bar\theta\big)\big(\chi_a(x)+im\theta_a u(x) \\ &-\sigma^\mu_{a\bar{a}}\bar\theta^{\bar{a}}\partial_\mu(u^K(x)+2i\theta\chi(x)-m\theta\theta u(x))\big). \end{aligned} \tag{4.3.3}$$

The superfield $\theta X(x,\theta,\bar{\theta})$ is equal to

$$\begin{aligned}\theta X(x,\theta,\bar{\theta}) =&(1-3i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu})\theta\chi(x)\\ &-(1-i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu})(\theta\sigma^{\nu}\bar{\theta}\partial_{\nu})u^{K}(x)\\ &+im\theta\theta u(x).\end{aligned} \tag{4.3.4}$$

By definition the superfield $X(x,\theta,\bar{\theta})$ is equal to

$$X(x,\theta,\bar{\theta}) \overset{def}{=} \theta X(x,\theta,\bar{\theta}) - \bar{\theta}\bar{X}(x,\theta,\bar{\theta}). \tag{4.3.5}$$

Similarly we can work out the representation-theoretical viewpoint here, but one has to remind oneself, that the ghost scalar and the ghost Majorana fields have both two scalar, respectively spinor, components. As a result they constitute a $\Omega_0 \otimes \Omega_0$-representation of the Supersymmetrygroup.

4.3.1 The Anti-Ghost Chiral Superfield

As for the free quantum fields we introduce an anti-ghost supefield which in the chiral case obeys the following (anti-)commutation relations:

Definition 4.3.2. The algebraic structure of the anti-ghost chiral superfield is determined by the (anti-)commutation relations

$$\begin{aligned}i\{Q_a,\tilde{u}(x)\} =&0, & i\{Q_a,\tilde{u}(x)^K\} \quad &=2i\tilde{\chi}_a(x),\\ i[Q_a,\tilde{\chi}_b(x)] =&im\epsilon_{ab}\tilde{u}(x), & i[Q_a,\bar{\tilde{\chi}}_{\bar{b}}(x)] \quad &=-\sigma^{\mu}_{a\bar{b}}\partial_{\mu}\tilde{u}(x).\end{aligned} \tag{4.3.6}$$

Please note that $\big(\tilde{\chi}_a(x)\big)^K = -\bar{\tilde{\chi}}_{\bar{a}}(x)$. Taking the adjungate relations of 4.3.2 therefore yield:

$$\begin{aligned}i\{\bar{Q}_{\bar{a}},\tilde{u}^K(x)\} =&0, & i\{\bar{Q}_{\bar{a}},\tilde{u}(x)\} \quad &=-2i\bar{\tilde{\chi}}_{\bar{a}}(x),\\ i[\bar{Q}_{\bar{a}},\bar{\tilde{\chi}}_{\bar{b}}(x)] =&im\epsilon_{\bar{a}\bar{b}}\tilde{u}^K(x), & i[\bar{Q}_{\bar{a}},\tilde{\chi}_b(x)] \quad &=\sigma^{\mu}_{b\bar{a}}\partial_{\mu}\tilde{u}^K(x).\end{aligned} \tag{4.3.7}$$

The computation of the Superfield $\tilde{U}(x,\theta,\bar{\theta})$ now proceeds as before and gives:

Result 3. *The anti-ghost chiral superfield is explicitly given by:*

$$\begin{aligned}\tilde{U}(x,\theta,\bar{\theta}) \overset{def}{=} & W(\theta,\bar{\theta})\tilde{u}(x)W^{-1}(\theta,\bar{\theta}) \\ = & \big(1 + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu} + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\big)\tilde{u}(x) \\ - & \big(1 - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\big)2i\bar{\theta}\bar{\tilde{\chi}}(x) + m\bar{\theta}\bar{\theta}\tilde{u}^K(x) \\ + & 2im\bar{\theta}\bar{\theta}\theta\tilde{\chi}(x),\end{aligned} \tag{4.3.8}$$

$$\begin{aligned}\tilde{X}_a(x,\theta,\bar{\theta}) \overset{def}{=} & W(\theta,\bar{\theta})\tilde{\chi}_a(x)W^{-1}(\theta,\bar{\theta}) \\ = & \big(1 - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu} + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\big)\big(\tilde{\chi}_a(x) - im\theta_a\tilde{u}(x) \\ - & \sigma^{\mu}_{a\bar{a}}\bar{\theta}^{\bar{a}}\partial_{\mu}(\tilde{u}^{\dagger}(x) + 2i\theta\tilde{\chi}(x) + m\theta\theta\tilde{u}(x))\big).\end{aligned} \tag{4.3.9}$$

The anti-superfield $\theta\tilde{X}(x,\theta,\bar{\theta})$ *is equal to*

$$\begin{aligned}\theta\tilde{X}(x,\theta,\bar{\theta}) = & (1 + 3i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu})\theta\tilde{\chi}(x) \\ + & (1 + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu})(\theta\sigma^{\nu}\bar{\theta}\partial_{\nu})\tilde{u}^K(x) \\ - & im\theta\theta\tilde{u}(x).\end{aligned} \tag{4.3.10}$$

By definition the superfield $\tilde{X}(x,\theta,\bar{\theta})$ *is equal to*

$$\tilde{X}(x,\theta,\bar{\theta}) \overset{def}{=} \theta\tilde{X}(x,\theta,\bar{\theta}) + \bar{\theta}\bar{\tilde{X}}(x,\theta,\bar{\theta}). \tag{4.3.11}$$

Again this happens to be a construction of the $\Omega_0 \otimes \Omega_0$-representation.

4.4 The Vector Superfield

Generally one calls a superfield a vector superfield if one of the components is a vector field. As shown in [14] one can construct a vector multiplet having scalar, spin 1/2 and vector components, but one encounters some difficulties when one applies our perturbative quantum gauge invariance. Here we are therefore going to construct another vector field where we start with a free Majorana field $\psi_a(x)$. We define:

Definition 4.4.1. Our vector superfield $V(x,\theta,\bar\theta)$ will be given by:

$$V(x,\theta,\bar\theta) \equiv W(\theta,\bar\theta)\big(\theta\psi(x)+\bar\theta\bar\psi(x)\big)W^{-1}(\theta,\bar\theta) \tag{4.4.1}$$

Lets work out the (anti)commutation relations between the operators Q_a, $\bar Q_{\bar b}$ and the various component fields.

The anti-commutator $\{Q_a, \bar\psi_{\bar b}(x)\}$ must be proportional to $\sigma^\mu_{a\bar b}\big(A_\mu(x)+iB_\mu(x)\big)$ because there is a one to one correspondence via the σ-matrices of bispinors of the type $(\frac{1}{2},\frac{1}{2})$ and four-vectors. We therefore write

$$i\{Q_a,\bar\psi_{\bar b}(x)\} = m\sigma^\mu_{a\bar b}\big(A_\mu(x)+iB_\mu(x)\big), \tag{4.4.2}$$

where the factor m is placed for dimensional reasons. Applying the Dirac equation we get:

$$\begin{aligned} i\{Q_a,\psi_b(x)\} =& \frac{i}{m}\{Q_a, i\sigma^\nu_{b\bar c}\partial_\nu\bar\psi^{\bar c}(x)\} \\ =& i\frac{\epsilon_{ad}}{m} i\sigma^\nu_{b\bar c}\partial_\nu\{Q^d,\bar\psi^{\bar c}(x)\} \\ =& \epsilon_{ad} i\sigma^\nu_{b\bar c}\partial_\nu\bar\sigma^{\mu\bar c d}\big(A_\mu(x)+iB_\mu(x)\big) \\ =& i(\sigma^\nu\bar\sigma^\mu)_{ba}\partial_\nu\big(A_\mu(x)+iB_\mu(x)\big). \end{aligned} \tag{4.4.3}$$

Taking adjoints on both sides gives:

$$i\{\bar Q_{\bar a},\bar\psi_{\bar b}(x)\} = i(\bar\sigma^\mu\sigma^\nu)_{\bar a\bar b}\partial_\nu\big(A_\mu(x)-iB_\mu(x)\big). \tag{4.4.4}$$

Next we have to fix the commutators $[Q_a, A_\mu(x)+iB_\mu(x)]$ and $[Q_a, A_\mu(x)-iB_\mu(x)]$. First note that $\big[Q_c,\{Q_a,\bar\psi_{\bar b}(x)\}\big]$, due to the Jacobi relations, must be antisymmetric in a and c. Therefore we conclude that $\big[Q_c,\{Q_a,\bar\psi_{\bar b}(x)\}\big] = \frac{1}{2}\epsilon_{ac}\bar\Lambda_{\bar b}(x)$ for some

Majorana field $\Lambda_a(x)$. Note that we may write $\bar{\Lambda}_{\bar{b}}(x) = \left[Q_c, \{Q^c, \bar{\psi}_{\bar{b}}(x)\}\right]$, from which follows immediately that $\{Q_a, \bar{\Lambda}_{\bar{b}}(x)\} = 0$. Using the Dirac equation we arrive at $\{Q_a, \Lambda_b(x)\} = 0$. Then, by the Jacobi relations one computes:

$$-\left[\bar{Q}_{\bar{a}}, \{Q_a, \Lambda_b(x)\}\right] = 0 = \left[Q_a, \{\bar{Q}_{\bar{a}}, \Lambda_b(x)\}\right] + \left[\Lambda_b(x), \{Q_{\bar{a}}, Q_a\}\right] = 2i\sigma^{\mu}_{a\bar{a}}\partial_\mu\Lambda_b(x), \tag{4.4.5}$$

from which one concludes that $\Lambda_a(x) = 0$, by the Dirac equation. We therefore arrive at:

$$\begin{aligned} [Q_c, A_\mu(x) + iB_\mu(x)] =& \frac{1}{m}\bar{\sigma}^{\bar{b}a}_\mu\left[Q_c, \{Q_a, \bar{\psi}_{\bar{b}}(x)\}\right] \\ =&0 \\ \Rightarrow&[Q_a, A_\mu(x)] = -i[Q_a, B_\mu(x)] \equiv -iP_{a\mu}(x). \end{aligned} \tag{4.4.6}$$

Here we encounter for the first time a mixed ”vector-spinor”-field. These have not been explicitely constructed in the first chapter, but by the reconstruction theorem, it will be enough to give its commutation relations (The transformation under an element of the Poincaré-group is rather evident). This will be done after we investigated the anti-commutation relations with the super-algebra.

To compute $\{Q_a, \bar{P}_{\mu\bar{b}}(x)\}$ we again use the Jacobi identity:

$$\begin{aligned} \{Q_a, \bar{P}_{\mu\bar{b}}(x)\} =& \frac{i}{2}\{Q_a, [\bar{Q}_{\bar{b}}, A_\mu(x) + iB_\mu(x)]\} \\ =& \frac{i}{2}\{\bar{Q}_{\bar{b}}, [A_\mu(x) + iB_\mu(x), Q_a]\} + \frac{i}{2}\left[\{Q_a, \bar{Q}_{\bar{b}}\}, A_\mu(x) + iB_\mu(x)\right] \\ =& i[\sigma^\nu_{a\bar{b}}P_\nu, A_\mu(x) + iB_\mu(x)] \\ =& \sigma^\nu_{a\bar{b}}\partial_\nu\big(A_\mu(x) + iB_\mu(x)\big). \end{aligned} \tag{4.4.7}$$

Finally, we need to know $\{Q_a, P_{\mu b}(x)\}$. Since we already know $\{Q_a, \bar{P}_{\mu\bar{b}}(x)\}$ it would be nice to have a Dirac-like equation of motion for $\bar{P}_{\mu\bar{b}}(x)$. Of course we have that $(\Box + m^2)\bar{P}_{\mu\bar{b}}(x) = 0$, since we work exclusively with free fields.

4.4.1 An Equation of Motion for $P_{\mu b}(x)$

From 4.4.2 we get

$$\big[Q_c, \{\bar{Q}_{\bar{a}}, \psi_b(x)\}\big] = im\sigma^{\mu}_{b\bar{a}}[Q_c, A_\mu(x) - iB_\mu(x)] = 2m\sigma^{\mu}_{b\bar{a}}P_{\mu c}. \tag{4.4.8}$$

But by Jacobi this is equal to

$$\big[\{\psi_b(x), Q_c\}, \bar{Q}_{\bar{a}}\big] + \big[\{Q_c, \bar{Q}_{\bar{a}}\}, \psi_b(x)\big], \tag{4.4.9}$$

which by 4.4.3 is equal to

$$(\sigma^\nu\bar{\sigma}^\mu)_{bc}\partial_\nu[A_\mu(x) + iB_\mu(x), \bar{Q}_{\bar{a}}] - 2i\sigma^{\mu}_{c\bar{a}}\partial_\mu\psi_b(x). \tag{4.4.10}$$

Using 4.4.6 this gives us a first inhomogeneous differential equation for $P_{\mu b}(x)$:

$$i(\sigma^\nu\bar{\sigma}^\mu)_{bc}\partial_\nu\bar{P}_{\mu\bar{a}}(x) - m\sigma^{\mu}_{b\bar{a}}P_{\mu c}(x) = i\sigma^{\mu}_{c\bar{a}}\partial_\mu\psi_b(x). \tag{4.4.11}$$

Let's multiply this equation by $\bar{\sigma}^{\alpha\bar{a}a}$, put $a = b$ and sum over b. We get:

$$\begin{aligned} -i(\sigma^\nu\bar{\sigma}^\mu\sigma^\alpha)_c^{\ \bar{a}}\partial_\nu\bar{P}_{\mu\bar{a}}(x) - 2mP^\alpha_c(x) =& i(\sigma^\mu\bar{\sigma}^\alpha)_c^{\ b}\partial_\mu\psi_b(x) \\ \Leftrightarrow P^\alpha_c(x) = \frac{-1}{2m}\big(i(\sigma^\mu\bar{\sigma}^\alpha)_c^{\ b}\partial_\mu\psi_b(x) +& i(\sigma^\nu\bar{\sigma}^\mu\sigma^\alpha)_c^{\ \bar{a}}\partial_\nu\bar{P}_{\mu\bar{a}}(x)\big). \end{aligned} \tag{4.4.12}$$

This equation allows us now to compute $\{Q_a, P^\alpha_c(x)\}$:

$$\begin{aligned} \{Q_a, P^\alpha_c(x)\} =& \frac{-1}{2m}\big((\sigma^\mu\bar{\sigma}^\alpha)_c^{\ b}\partial_\mu i\{Q_a, \psi_b(x)\} + (\sigma^\nu\bar{\sigma}^\mu\sigma^\alpha)_c^{\ \bar{a}}\partial_\nu i\{Q_a, \bar{P}_{\mu\bar{a}}(x)\}\big) \\ \overset{4.4.3,\ 4.4.7}{=}& \frac{-1}{2m}\big((\sigma^\mu\bar{\sigma}^\alpha\sigma^\nu\bar{\sigma}^\beta)_{ca} i\partial_\mu\partial_\nu\big(A_\beta(x) + iB_\beta(x)\big) \\ & - (\sigma^\nu\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta)_{ca} i\partial_\nu\partial_\beta\big(A_\mu(x) + iB_\mu(x)\big)\big) \\ =& \frac{i}{m}\epsilon_{ac}\big(\Box\big(A^\alpha(x) + iB^\alpha(x)\big) - 2\partial^\alpha\partial_\mu\big(A^\mu(x) + iB^\mu(x)\big)\big) \\ =& i\epsilon_{ca}\Big(m\big(A^\alpha(x) + iB^\alpha(x)\big) + \frac{2}{m}\partial^\alpha\partial_\mu\big(A^\mu(x) + iB^\mu(x)\big)\Big). \end{aligned} \tag{4.4.13}$$

Remark 4.4.1. It has to be stressed that the fields $\psi_a(x)$ and $P_a^\mu(x)$ are not independent. In fact $\psi_a(x)$ is "hidden" in the field $P_a^\mu(x)$, as from 4.4.11 it follows that:

$$\begin{aligned} i\sigma^\mu_{c\bar{a}}\partial_\mu\psi^c(x) =& i(\sigma^\nu\bar{\sigma}^\mu)_c{}^c\partial_\nu\bar{P}_{\mu\bar{a}}(x) + m\sigma^\mu_{c\bar{a}}P^c_\mu \\ \Rightarrow m\bar{\psi}_{\bar{a}}(x) =& -2i\partial\bar{P}_{\bar{a}} - m\sigma^\mu_{c\bar{a}}P^c_\mu \\ \Rightarrow \psi_a(x) =& \frac{2i}{m}\partial P_a - \sigma^\mu_{a\bar{c}}\bar{P}^{\bar{c}}_\mu(x). \end{aligned} \tag{4.4.14}$$

Now we know all the (anti)commutation relations of our component fields with the super-algebra.

To compute the superfield we can proceed as in the case of the chiral superfield. Let us compute

$$\begin{aligned} V_a(x,\theta,\bar{\theta}) \stackrel{\text{def}}{=}& W(\theta,\bar{\theta})\psi_a(x)W^{-1}(\theta,\bar{\theta}) \\ =& e^{i\theta Q}e^{i\bar{\theta}\bar{Q}}e^{\theta\sigma^\mu\bar{\theta}P_\mu}\psi_a(x)e^{-\theta\sigma^\mu\bar{\theta}P_\mu}e^{-i\bar{\theta}\bar{Q}}e^{-i\theta Q} \\ \stackrel{\text{def}}{=}& e^{i\theta Q}e^{i\bar{\theta}\bar{Q}}\psi_a(y)e^{-i\bar{\theta}\bar{Q}}e^{-i\theta Q}, \end{aligned} \tag{4.4.15}$$

where

$$\begin{aligned} \psi_a(y) =& e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu}\psi_a(x) \\ =& \psi_a(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\psi_a(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\psi_a(x) \\ =& (1 + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta} - i\theta\sigma^\mu\bar{\theta}\partial_\mu)\psi_a(x). \end{aligned} \tag{4.4.16}$$

next we need to know $e^{i\bar{\theta}\bar{Q}}\psi_a(x)e^{-i\bar{\theta}\bar{Q}}$:

$$\begin{aligned} e^{i\bar{\theta}\bar{Q}}\psi_a(x)e^{-i\bar{\theta}\bar{Q}} =& \psi_a(x) + i[\bar{\theta}\bar{Q},\psi_a(x)] + \frac{1}{2}[i\bar{\theta}\bar{Q},[i\bar{\theta}\bar{Q},\psi_a(x)]] \\ =& \psi_a(x) - i\bar{\theta}^{\bar{b}}\{\bar{Q}_{\bar{b}},\psi_a(x)\} - \frac{1}{2}[i\bar{\theta}\bar{Q}, i\bar{\theta}^{\bar{b}}\{\bar{Q}_{\bar{b}},\psi_a(x)\}] \\ =& \psi_a(x) + m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\big(A_\mu(x) - iB_\mu(x)\big). \end{aligned} \tag{4.4.17}$$

Similarly one finds

$$\begin{aligned}
e^{i\theta Q}\psi_a(x)e^{-i\theta Q} =&\psi_a(x) + i\theta^b(\sigma^\nu\bar{\sigma}^\mu)_{ab}\partial_\nu\big(A_\mu(x) + iB_\mu(x)\big);\\
e^{i\theta Q}\big(A_\mu(x) - iB_\mu(x)\big)e^{-i\theta Q} =&A_\mu(x) - iB_\mu(x) + 2\theta P_\mu(x)\\
&-\theta\theta\Big(m\big(A_\mu(x) + iB_\mu(x)\big) + \frac{2}{m}\partial_\mu\partial_\nu\big(A^\nu(x) + iB^\nu(x)\big)\Big).
\end{aligned} \tag{4.4.18}$$

Putting all together we obtain the

Result 4. *Our vector superfield $V(x,\theta,\bar{\theta})$ satisfies the relations:*

$$\begin{aligned}
i\{Q_a, \bar{\psi}_{\bar{b}}(x)\} =&m\sigma^\mu_{a\bar{b}}\big(A_\mu(x) + iB_\mu(x)\big),\\
i\{Q_a, \psi_b(x)\} =&i(\sigma^\nu\bar{\sigma}^\mu)_{ba}\partial_\nu\big(A_\mu(x) + iB_\mu(x)\big);\\
i[Q_a, A_\mu(x) + iB_\mu(x)] =&0,\\
i[Q_a, A_\mu(x) - iB_\mu(x)] =&2P_{\mu a};\\
i\{Q_a, P^\nu_b(x)\} =&\epsilon_{ab}\Big(m\big(A^\nu(x) + iB^\nu(x)\big) + \frac{2}{m}\partial^\nu\partial_\mu\big(A^\mu(x) + iB^\mu(x)\big)\Big),\\
i\{Q_a, \bar{P}^\mu_{\bar{b}}(x)\} =&i\sigma^\nu_{a\bar{b}}\partial_\nu\big(A^\mu(x) + iB^\mu(x)\big).
\end{aligned} \tag{4.4.19}$$

Writing $c_\mu(x)$ for $A^\nu(x) + iB^\nu(x)$ one has

$$\begin{aligned}
V_a(x,\theta,\bar{\theta}) =&(1 + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta} - i\theta\sigma^\mu\bar{\theta}\partial_\mu)\psi_a(x)\\
&+(1 - i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)i\theta^b(\sigma^\nu\bar{\sigma}^\mu)_{ab}\partial_\nu c_\mu(x)\\
&-m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x)\\
&+(1 - i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}(c^K_\mu(x) + 2\theta P_\mu(x)),\\
\theta^a V_a(x,\theta,\bar{\theta}) =&(1 - i\theta\sigma^\mu\bar{\theta}\partial_\mu)\theta\psi(x) + i\theta(\sigma^\nu\bar{\sigma}^\mu)\theta\partial_\nu c_\mu(x)\\
&+(1 - i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)m\theta\sigma^\mu\bar{\theta}c^K_\mu(x) + 2m\theta\sigma^\mu\bar{\theta}\theta P_\mu(x),\\
V(x,\theta,\bar{\theta}) =&\theta^a V_a(x,\theta,\bar{\theta}) + \bar{\theta}_{\bar{a}}\bar{V}^{\bar{a}}(x,\theta,\bar{\theta}).
\end{aligned} \tag{4.4.20}$$

Starting with the component $c_\mu(x)$ the superfield reads:

$$\begin{aligned}V_\mu(x,\theta,\bar\theta) \overset{def}{=} & W(\theta,\bar\theta)c_\mu(x)W^{-1}(\theta,\bar\theta) \\ =&(1+\frac{m^2}{4}\theta\theta\bar\theta\bar\theta + i\theta\sigma^\nu\bar\theta\partial_\nu)c_\mu(x) \\ +&(1-i\theta\sigma^\nu\bar\theta\partial_\nu)2\bar\theta\bar P_\mu(x) \\ -&m\bar\theta\bar\theta(\delta^\nu_\mu+\frac{2}{m^2}\partial_\mu\partial^\nu)\big(c^K_\nu(x)+2\theta P_\nu(x)\big).\end{aligned} \tag{4.4.21}$$

Here the representation point of view is a little bit more involved.

Starting with the highest spin-value we get:

$$\begin{aligned} u(\frac{3}{2},\frac{3}{2}) &\xrightarrow{Q_1} u_A(1,1) \quad \xrightarrow{\bar Q_{\bar 2}} u(\frac{3}{2},\frac{1}{2}), \\ u(\frac{3}{2},\frac{3}{2}) &\xrightarrow{\bar Q_{\bar 2}} u_B(1,1) \quad \xrightarrow{Q_1} u(\frac{3}{2},\frac{1}{2}),\end{aligned} \tag{4.4.22}$$

as well as

$$\begin{aligned} u(\frac{3}{2},-\frac{3}{2}) &\xrightarrow{\bar Q_{\bar 1}} u_B(1,-1) \quad \xrightarrow{Q_2} u(\frac{3}{2},\frac{-1}{2}), \\ u(\frac{3}{2},-\frac{3}{2}) &\xrightarrow{Q_2} u_A(1,-1) \quad \xrightarrow{\bar Q_{\bar 1}} u(\frac{3}{2},\frac{-1}{2}).\end{aligned} \tag{4.4.23}$$

But a spin-$\frac{3}{2}$-multiplet has another spin-$\frac{1}{2}$-state, $u(\frac{1}{2}),\frac{1}{2}$ and a spin-1-multiplet has a state $u(1,0)$. We therefore get an additional representation-structure:

$$\begin{aligned} u(\frac{1}{2},\frac{1}{2}) &\xrightarrow{Q_1} u_A(1,0) \quad \xrightarrow{\bar Q_{\bar 2}} u(\frac{1}{2},-\frac{1}{2}), \\ u(\frac{1}{2},\frac{1}{2}) &\xrightarrow{\bar Q_{\bar 2}} u_B(1,0) \quad \xrightarrow{Q_1} u(\frac{1}{2},-\frac{1}{2}).\end{aligned} \tag{4.4.24}$$

This is known as the Ω_1-representation.

But, there are two other scalar particles generated by the divergence of the two vector fields, and they constitute a Super-multiplet with the divergence of spin-$\frac{3}{2}$-field. That these two divergences come together in the same Super-multiplet can be seen through the commutation-rule

$$i[Q_a,\partial A - i\partial B] \overset{4.4.19}{=} 2\partial P_a, \tag{4.4.25}$$

and similarly for the adjoint $\bar Q_{\bar b}$. Therefore, we get an additional Ω_0-representation. All in all, the vector Superfield builds a $\Omega_1\otimes\Omega_0$-representation.

4.5 The Commutators

As already mentioned in the first chapter, a quantum field is uniquely determined by its Wightman functions. For a free field it suffices to give their commutation relations. For free superfields it is still true. In this section we therefore calculate the commutators of the various superfields.

One can of course take the expression of the superfields in their components and carry through a straightforward computation. This is awfully lengthy and there is a more elegant way. To this end we will need the following

Theorem 4.5.1. *Let $\varphi(x)$ and $\psi(x)$ be a free bosonic, respectively a free fermionic quantum field and let $\Phi(\theta,\bar\theta,x)$ and $\Psi(\theta,\bar\theta,x)$ be their supersymmetric extensions (Here we omit the explicit tensorial nature of the fields). Then one has:*

$$[\Phi(\theta,\bar\theta,x),\Phi(\Gamma,\bar\Gamma,y)] = \exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)[\varphi(x),\Phi(\Gamma-\theta,\bar\Gamma-\bar\theta,y)] \tag{4.5.1}$$

and

$$\{\Psi(\theta,\bar\theta,x),\Psi(\Gamma,\bar\Gamma,y)\} = \exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)\{\psi(x),\Psi(\Gamma-\theta,\bar\Gamma-\bar\theta,y)\} \tag{4.5.2}$$

Proof. We prove the case of bosonic fields. The fermionic case is treated exactly in the same way, replacing commutators by anticommutators mutandis mutandi.

Since the superfields $\Phi(x,\theta,\bar\theta)$ and $\Phi(y,\Gamma,\bar\Gamma)$ are a sum of products of Grassmanian variables with free quantum fields and their derivatives the commutator will be a sum of products of Grassmanian variables with the Jordan-Pauli distribution $D_m(x-y)$ and its derivatives, the whole multiplying the identity operator $\mathbb{1}$ in Fock-space.

The operator $\theta Q + \bar\theta\bar Q$ commutes with all Grassmanian variables, since the Grassmanian variables anticommute among themselves and since the operators Q_a and

$\bar{Q}_{\bar{b}}$ anticommute with any Grassmanian variable. As a consequence, the operator $\exp(i\theta Q + i\bar{\theta}\bar{Q}) = W(\theta,\bar{\theta})$ commutes with the supercommutator. Therefore one has:

$$\begin{aligned}
[\Phi(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)] =& W^{-1}(\theta,\bar{\theta})[\Phi(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)]W(\theta,\bar{\theta}) \\
=& W^{-1}(\theta,\bar{\theta})[W(\theta,\bar{\theta})\varphi(x)W^{-1}(\theta,\bar{\theta}),\Phi(\Gamma,\bar{\Gamma},y)]W(\theta,\bar{\theta}) \\
=& W^{-1}(\theta,\bar{\theta})W(\theta,\bar{\theta})[\varphi(x),W^{-1}(\theta,\bar{\theta})\Phi(\Gamma,\bar{\Gamma},y)W(\theta,\bar{\theta})]W^{-1}(\theta,\bar{\theta})W(\theta,\bar{\theta}) \\
=& [\varphi(x),W^{-1}(\theta,\bar{\theta})W(\Gamma,\bar{\Gamma})\varphi(y)W^{-1}(\Gamma,\bar{\Gamma})W(\theta,\bar{\theta})]
\end{aligned} \tag{4.5.3}$$

But now

$$\begin{aligned}
W^{-1}(\theta,\bar{\theta})W(\Gamma,\bar{\Gamma}) &= \exp(-i\theta Q - i\bar{\theta}\bar{Q})\exp(i\Gamma Q + i\bar{\Gamma}\bar{Q}) \\
&= \exp(\theta\sigma^{\mu}\bar{\Gamma}P_{\mu} - \Gamma\sigma^{\nu}\bar{\theta}P_{\nu})\exp(i(\Gamma-\theta)Q + i(\bar{\Gamma}-\bar{\theta})\bar{Q}) \\
&= \exp(\theta\sigma^{\mu}\bar{\Gamma}P_{\mu} - \Gamma\sigma^{\nu}\bar{\theta}P_{\nu})W(\Gamma-\theta,\bar{\Gamma}-\bar{\theta}).
\end{aligned} \tag{4.5.4}$$

We therefore get

$$[\Phi(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)] = \exp(i\theta\sigma^{\mu}\bar{\Gamma}\partial_{\mu} - i\Gamma\sigma^{\nu}\bar{\theta}\partial_{\nu})[\varphi(x),\Phi(\Gamma-\theta,\bar{\Gamma}-\bar{\theta},y)]. \tag{4.5.5}$$

□

This is a much better looking formula to compute the super-commutators. All we need to know now are the (anti)commutators between the various free quantum fields composing the superfield.

Applied on the chiral scalar superfield we obtain, using 4.2.16:

$$\begin{aligned}
[\Phi(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)] =& -2i\exp(i\theta\sigma^{\mu}\bar{\Gamma}\partial_{\mu} - i\Gamma\sigma^{\nu}\bar{\theta}\partial_{\nu})(\bar{\Gamma}-\bar{\theta})^2 D_m(x-y) \\
=& (\bar{\Gamma}-\bar{\theta})^2 D_m(x-y), \text{ because } \bar{\Gamma}^{\bar{a}}(\bar{\Gamma}-\bar{\theta})^2 = \bar{\theta}^{\bar{a}}(\bar{\Gamma}-\bar{\theta})^2 = 0; \\
[\Phi^{\dagger}(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)] =& 2i\exp(i\theta\sigma^{\mu}\bar{\Gamma}\partial_{\mu} - i\Gamma\sigma^{\nu}\bar{\theta}\partial_{\nu}) \\
&\times\exp(i(\Gamma-\theta)\sigma^{\alpha}(\bar{\Gamma}-\bar{\theta})\partial_{\alpha})D_m(x-y)
\end{aligned} \tag{4.5.6}$$

The Dirac equation, lemma 4.2.1 and the equations in definition 4.2.2 allow one to compute also the (anti)commutators $[\Psi(\theta,\bar{\theta},x),\Phi(\Gamma,\bar{\Gamma},y)]$ and $\{\Psi(\theta,\bar{\theta},x),\Psi(\Gamma,\bar{\Gamma},y)\}$. Next we have for the chiral ghost and anti-ghost superfields:

$$\begin{aligned}[X_a(\theta,\bar{\theta},x),\tilde{X}_b(\Gamma,\bar{\Gamma},y)] =& \exp(i\theta\sigma^\mu\bar{\Gamma}\partial_\mu - i\Gamma\sigma^\nu\bar{\theta}\partial_\nu)\exp(i(\Gamma-\theta)\sigma^\alpha(\bar{\Gamma}-\bar{\theta})\partial_\alpha) \\ &\times(im\epsilon_{ab} + 2m\sigma^\beta_{b\bar{c}}(\Gamma_a-\theta_a)(\bar{\Gamma}^{\bar{c}}-\bar{\theta}^{\bar{c}})\partial_\beta)D_m(x-y);\end{aligned} \tag{4.5.7}$$

For the vector superfield computations get more involved. Firstly, note that:

$$\begin{aligned}[A_\mu(x)+iB_\mu(x),P_{a\nu}(y)] \overset{4.4.19}{=}& \frac{i}{2}[A_\mu(x)+iB_\mu(x),[Q_a,A_\mu(y)-iB_\mu(y)]] \\ =&\frac{-i}{2}[Q_a,[A_\mu(y)-iB_\mu(y),A_\mu(x)+iB_\mu(x)]] \\ +&\frac{-i}{2}[A_\mu(x)-iB_\mu(x),[A_\mu(y)+iB_\mu(y),Q_a]] \\ =&0+0=0,\end{aligned} \tag{4.5.8}$$

the first commutator beeing zero because $[A_\mu(y)-iB_\mu(y),A_\mu(x)+iB_\mu(x)]$ is a $\mathbb{1}$-valued distribution and commutes with any operator, in particular with Q_a and the second one beeing zero because of the third equation in 4.4.19.

Taking adjoints one has

$$[A_\mu(x)-iB_\mu(x),\bar{P}_{\bar{a}\nu}(y)] = 0. \tag{4.5.9}$$

In appendix A we prove that 4.4.11 and 4.4.14 implies

$$m(\sigma^\mu\bar{\sigma}^\alpha)_b{}^c\partial_\alpha P_{\mu c}(x) = im^2\psi_b(x) \tag{4.5.10}$$

This equation then implies that

$$[A_\mu(x)+iB_\mu(x),\psi_b(y)] = 0. \tag{4.5.11}$$

Applying the Majorana equation and taking adjoints gives

$$[A_\mu(x)-iB_\mu(x),\psi_b(y)] = 0. \tag{4.5.12}$$

As a consequence of equation 4.4.12 we get

$$[P_c^\alpha(x), A_\mu(y) - iB_\mu(y)] = 0. \tag{4.5.13}$$

We can now compute the anti-commutation relations of the mixed field $P_\mu^a(x)$. Using the Jacobi-identity we have:

$$\begin{aligned}
\{P_a^\mu(x), P_b^\nu(y)\} =&\{i[Q_a, A^\mu(x)], P_b^\nu(y)\}\\
=&\{iQ_a, [A^\mu(x), P_b^\nu(y)]\} - i[A^\mu(x), \{Q_a, P_b^\nu(y)\}]\\
=&-[A^\mu(x), m\epsilon_{ab}(\delta_\alpha^\nu + \frac{2}{m^2}\partial^\nu\partial_\alpha)(A^\alpha(y) + iB^\alpha(y))]\\
=&-m(\tau)^2\epsilon_{ab}(\delta_\alpha^\nu + \frac{2}{m^2}\partial^\nu\partial_\alpha)[A^\mu(x), A^\alpha(y)]\\
=&-im(\tau)^2\epsilon_{ab}(g^{\mu\nu} + \frac{2}{m^2}\partial^\nu\partial^\mu)D_m(x-y),
\end{aligned} \tag{4.5.14}$$

and similarly

$$\begin{aligned}
\{P_a^\mu(x), \bar{P}_{\bar{b}}^\nu(y)\} =&\{i[Q_a, A^\mu(x)], \bar{P}_{\bar{b}}^\nu(y)\}\\
=&\{iQ_a, [A^\mu(x), \bar{P}_{\bar{b}}^\nu(y)]\} - i[A^\mu(x), \{Q_a, \bar{P}_{\bar{b}}^\nu(y)\}]\\
=&-i\sigma_{a\bar{b}}^\alpha\partial_\alpha^y[A^\mu(x), A^\nu(y) + iB^\nu(y)]\\
=&-(\tau)^2\sigma_{a\bar{b}}^\alpha\partial_\alpha g^{\nu\mu}D_m(x-y).
\end{aligned} \tag{4.5.15}$$

Here we take the general commutation relation

$$[A^\mu(x), A^\nu(y)] = i(\tau)^2 g^{\nu\mu}D_m(x-y), \tag{4.5.16}$$

with $\tau \in \mathbb{R}$.

Since remark 4.4.1 shows that $\psi_a(x)$ is "hidden" in the field $P_b^\mu(x)$, we expect these fields to have non-trivial anti-commutators. Indeed:

$$\begin{aligned}
\{\psi_a(x), P_b^\mu(y)\} \overset{4.4.14}{=}&\{\frac{2i}{m}\partial P_a(x) - \sigma_{a\bar{c}}^\nu\bar{P}_\nu^{\bar{c}}(x), P_b^\mu(y)\}\\
=&\frac{2i}{m}\partial_\nu^x\{P_a^\nu(x), P_b^\mu(y)\} - \sigma_{a\bar{c}}^\nu\{\bar{P}_\nu^{\bar{c}}(x), P_b^\mu(y)\}\\
=&-2(\tau)^2\epsilon_{ab}\partial^\mu D_m(x-y) + (\tau)^2\sigma_{a\bar{c}}^\mu\epsilon^{\bar{c}\bar{b}}\sigma_{b\bar{b}}^\nu\partial_\nu D_m(x-y)\\
=&-2(\tau)^2\epsilon_{ab}\partial^\mu D_m(x-y) + (\tau)^2(\sigma^\mu\bar{\sigma}^\nu)_{ab}\partial_\nu D_m(x-y).
\end{aligned} \tag{4.5.17}$$

Applying the Majorana-equation gives

$$
\begin{aligned}
\{\bar{\psi}_{\bar{c}}(x), P^{\mu}_{b}(y)\} =& \frac{-i}{m}\sigma^{\rho}_{a\bar{c}}\partial_{\rho}\{\psi^{a}(x), P^{\mu}_{b}(y)\}\\
=& \frac{i(\tau)^2}{m}\sigma^{\rho}_{a\bar{c}}\partial_{\rho}\big(2\delta^{a}_{\ b}\partial^{\mu}D_m(x-y) - (\sigma^{\mu}\bar{\sigma}^{\nu})^{a}_{\ b}\partial_{\nu}D_m(x-y)\big)\\
=& \frac{2i(\tau)^2}{m}\sigma^{\rho}_{b\bar{c}}\partial_{\rho}\partial^{\mu}D_m(x-y) + \frac{i(\tau)^2}{m}(\sigma^{\nu}\bar{\sigma}^{\mu}\sigma^{\rho})_{b\bar{c}}\partial^{2}_{\rho\nu}D_m(x-y)\\
=& \frac{4i(\tau)^2}{m}\sigma^{\rho}_{b\bar{c}}\partial_{\rho}\partial^{\mu}D_m(x-y) + i(\tau)^2 m\sigma^{\mu}_{b\bar{c}}D_m(x-y).
\end{aligned}
\tag{4.5.18}
$$

Finally, the super(anti)commutators for the vector field reed:

$$
\begin{aligned}
[V_{\mu}(\theta,\bar{\theta},x), V_{\nu}(\Gamma,\bar{\Gamma},y)] =& 2i(\tau)^2 m\exp(i\theta\sigma^{\mu}\bar{\Gamma}\partial_{\mu} - i\Gamma\sigma^{\nu}\bar{\theta}\partial_{\nu})(\bar{\Gamma}-\bar{\theta})^2(g_{\mu\nu} + \frac{2}{m^2}\partial_{\mu}\partial_{\nu})D_m(x-y)\\
=& 2i(\tau)^2 m(\bar{\Gamma}-\bar{\theta})^2(g_{\mu\nu} + \frac{2}{m^2}\partial_{\mu}\partial_{\nu})D_m(x-y),\\
& \text{because } \bar{\Gamma}^{\bar{a}}(\bar{\Gamma}-\bar{\theta})^2 = \bar{\theta}^{\bar{a}}(\bar{\Gamma}-\bar{\theta})^2 = 0;\\
[V^{K}_{\mu}(\theta,\bar{\theta},x), V_{\nu}(\Gamma,\bar{\Gamma},y)] =& -2i(\tau)^2\exp(i\theta\sigma^{\mu}\bar{\Gamma}\partial_{\mu} - i\Gamma\sigma^{\nu}\bar{\theta}\partial_{\nu})\\
& \times\exp(-i(\Gamma-\theta)\sigma^{\alpha}(\bar{\Gamma}-\bar{\theta})\partial_{\alpha})g_{\mu\nu}D_m(x-y)
\end{aligned}
\tag{4.5.19}
$$

Similarly we have

$$\begin{aligned}
\{V_a(\theta,\bar\theta,x),V_b(\Gamma,\bar\Gamma,y)\} =&\exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)\{\psi_a(x),V_b(\Gamma-\theta,\bar\Gamma-\bar\theta,y)\}\\
=&(\tau)^2\exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)\exp(i(\Gamma-\theta)\sigma^\mu(\bar\Gamma-\bar\theta)\partial_\mu)\\
&\times\big(im\epsilon_{ab} - 4m\sigma^\alpha_{b\bar b}(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)_a\partial_\alpha\\
&+m\sigma^\alpha_{b\bar b}(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)^c g_{\alpha\beta}(\sigma^\beta\bar\sigma^\gamma)_{ac}\partial_\gamma\big)D_m(x-y)\\
=&(\tau)^2\exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)\exp(i(\Gamma-\theta)\sigma^\mu(\bar\Gamma-\bar\theta)\partial_\mu)\\
&\times\big(im\epsilon_{ab} - 4m\sigma^\alpha_{b\bar b}(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)_a\partial_\alpha\\
&+4m(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)^c(\sigma^\gamma_{b\bar b}\epsilon_{ca} - \sigma^\gamma_{a\bar b}\epsilon_{cb})\partial_\gamma\big)D_m(x-y)\\
=&(\tau)^2\exp(i\theta\sigma^\mu\bar\Gamma\partial_\mu - i\Gamma\sigma^\nu\bar\theta\partial_\nu)\exp(i(\Gamma-\theta)\sigma^\mu(\bar\Gamma-\bar\theta)\partial_\mu)\\
&\times\big(im\epsilon_{ab} - 8m\sigma^\alpha_{b\bar b}(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)_a\partial_\alpha\\
&+4m(\bar\Gamma-\bar\theta)^{\bar b}(\Gamma-\theta)_b\sigma^\gamma_{a\bar b}\partial_\gamma\big)D_m(x-y)
\end{aligned} \tag{4.5.20}$$

In the most general case one may write:

$$[c_\mu(x),c^K_\nu(y)] = 2i((\tau_0)^2 g_{\mu\nu} - |\vec\tau|^2\partial_\mu\partial_\nu)D_m(x-y), \tag{4.5.21}$$

where we write $c_\mu(x)$ for $A_\mu(x)+iB_\mu(x)$. But this commutation relations are ruled out by the condition of normalizability of the theory.

Remark 4.5.1. One advantage that one gets in supersymmetric field theory is, that the splitting-procedure for the construction of the S-matrix described in chapter 2 behaves mathematically much better than in the usual quantum field theories. Indeed, usually one has to split distributions like the Jordan-Pauli-function $D_m(x-y)$, which has as singular order $\omega=-2$. For this distribution the splitting problem is simple, but as soon as one acts with a derivation on a distribution, its singular order increases by 1. One also encounters difficult situations already in QED, where so-called "loop-graphs" have singular order $\omega = 2$. But as soon as $\omega > 1$, one has to be very careful with the splitting procedure.

In the supersymmetric case the T_i's appearing in the causal construction of the S-matrix are distributions on the super-space $\mathcal{M} \times (\theta, \bar{\theta})$. They therefore are also polynomial expressions in the variables θ and $\bar{\theta}$. When one is computing the integrals $\int \ldots d^4x d^2\theta d^2\bar{\theta}$ of such distributions, only those multiplied by the quantities $\theta\theta\bar{\theta}\bar{\theta}$ survive. This is because an integration in the Grassman-variables θ behaves as a derivation in the latter.

As a result, many graphes, espacially loop-graphes, which are divergent when integrated over d^4x cancel out, because they happen to be multiplied by a second or third power only in the Grassman-numbers θ or $\bar{\theta}$.

Chapter 5

Construction of the super-gauge charge

5.1 Introduction

We now arrive to the main part of this work, which is the construction of the gauge charge Q. As in the usual perturbative gauge invariant quantum field theory [22] one would like to have a nilpotent gauge charge Q, $Q^2 = 0$, which determines the physical subspace of the theory.

5.2 Commutation Relations With The Component Fields

The vector superfield has a vector component $A_\mu + iB_\mu$. As in usual quantum field theory we would like the gauge charge to fulfill the following relations:

$$\begin{aligned} [Q, A_\mu + iB_\mu] =& \partial_\mu u, \\ [Q, A_\mu - iB_\mu] =& -\partial_\mu u^K, \end{aligned} \tag{5.2.1}$$

where u and u^K are respectively a ghost scalar field and its conjugate. These we want to be the component of a scalar ghost superfield. Therefore we also introduce a Majorana ghost field $\chi_a(x)$. Altogether these ghost-fields will constitute a scalar superghost field, where we expect Q_a to have anticommutation relations with $u(x)$, since $u(x)$ is quantized like a spinor-field. Similarly we want $\chi_a(x)$ to have commutation relations with Q_b. To continue this discussion we impose the

Assumption 1. *The gauge charge Q is self-conjugate $Q^K = Q$*

The self-conjugateness of the gauge charge is a natural assumption if one wants it to define the physical subspace of the theory (see the discussion at the end of the subsection concerning the free vector quantum fields).
The spinor component ψ_a of the super-vector field also has a gauge variation. From the equalities

$$\begin{aligned} \big[Q_b, \{Q, \psi_a\}\big] =& \big[\{Q_b, \psi_a\}, Q\big] + \big[\{Q, Q_b\}, \psi_a\big] \\ =& (\sigma^\nu \bar{\sigma}^\mu)_{ab} \partial_\nu [A_\mu + iB_\mu, Q] \\ =& -(\sigma^\nu \bar{\sigma}^\mu)_{ab} \partial_\nu \partial_\mu u \\ =& -\Box \epsilon_{ba} u = m^2 \epsilon_{ba} u, \end{aligned} \tag{5.2.2}$$

one can conclude, using $[Q_b, \chi_a] = -m\epsilon_{ba}u$, that

$$\{Q, \psi_a\} = -m\chi_a. \tag{5.2.3}$$

The gauge variation for the spin-3/2 field P_a^μ follows from

$$\begin{aligned} \{Q, P_a^\mu\} =& \frac{i}{2}\{Q, [Q_a, A^\mu - iB^\mu]\} \\ =& \frac{i}{2}\{Q_a, [A^\mu - iB^\mu, Q]\} + \frac{i}{2}\big[\{Q, Q_a\}, A^\mu - iB^\mu\big] \\ =& \frac{i}{2}\partial^\mu \{Q_a, u^K\} = i\partial^\mu \chi_a. \end{aligned} \tag{5.2.4}$$

Therefore, assuming the usual gauge variation for the vector component of the superfield as well as the requirement that the gauge variation of the vector superfield gives a ghost superfield we arrive to the

Assumption 2. *The gauge variation of the components of the vector superfield are:*

$$\begin{aligned} \{Q,\psi_a\} =& \quad -m\chi_a, \quad \{Q,\bar{\psi}_{\bar{a}}\} = \quad -m\bar{\chi}_{\bar{a}}; \\ [Q,A^\mu + iB^\mu] =& \;\; \partial^\mu u, \;\; [Q,A^\mu - iB^\mu] = \;\; -\partial^\mu u^K; \\ \{Q,P_a^\mu\} =& \quad i\partial^\mu\chi_a, \;\; \{Q,\bar{P}_{\bar{a}}^\mu\} = \;\; -i\partial^\mu\bar{\chi}_{\bar{a}}. \end{aligned} \tag{5.2.5}$$

The equations on the right just follow from those on the left by taking the conjugate relations.

We can now explicitly check the gauge variation of the vector superfield:

$$\begin{aligned} [Q,\theta^a V_a(x,\theta,\bar{\theta})] =&(1 - i\theta\sigma^\mu\bar{\theta}\partial_\mu)[Q,\theta\psi(x)] + i\theta(\sigma^\nu\bar{\sigma}^\mu)\theta\partial_\nu[Q,c_\mu(x)] \\ &+(1 + i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)m\theta\sigma^\mu\bar{\theta}[Q,c_\mu^K(x)] + 2m\theta\sigma^\mu\bar{\theta}[Q,\theta P_\mu(x)] \\ =&m(1 - i\theta\sigma^\mu\bar{\theta}\partial_\mu)\theta\chi(x) + i\theta(\sigma^\nu\bar{\sigma}^\mu)\theta\partial_\nu\partial_\mu u(x) \\ &-(1 + i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)m\theta\sigma^\mu\bar{\theta}\partial_\mu u^K(x) - 2im\theta\sigma^\mu\bar{\theta}\partial_\mu\theta\chi(x) \\ =&m(1 - 3i\theta\sigma^\mu\bar{\theta}\partial_\mu)\theta\chi(x) + i\theta\theta m^2 u(x) \\ &-(1 + i\theta\sigma^\alpha\bar{\theta}\partial_\alpha)m\theta\sigma^\mu\bar{\theta}\partial_\mu u^K(x) \\ =&m\theta^a X_a(x,\theta,\bar{\theta}). \end{aligned} \tag{5.2.6}$$

The gauge variation of the components of the ghost superfield follow from the Jacobi relations. Using the results from the last assumption we have:

$$\begin{aligned} [Q,\chi_a] =&\frac{i}{m}\big[Q,\{Q,\psi_a\}\big] \\ =&\frac{i}{m}\big[\{Q,\psi_a\},Q\big] + \frac{i}{m}\big[\{Q,Q\},\psi\big] \\ =&-\frac{i}{m}\big[Q,\{Q,\psi_a\}\big] \text{ (because } Q^2 = 0) \\ =&i[Q,\chi_a]. \end{aligned} \tag{5.2.7}$$

Therefore the gauge variation of χ_a must vanish. A similar computation shows that $\bar{\chi}_{\bar{b}}$, u and u^K have no gauge variation neither:

Result 5. *The ghost superfield has a vanishing gauge variation.*

This result is consistent with the (anti)commutation relations of a ghost field. Indeed we have:

$$\begin{aligned}
[\chi_a(x), \chi_b(y)] =& \frac{i}{m}[\chi_a(x), \{Q, \psi_b(y)\}] \\
=& \frac{i}{m}\{[\psi_b(y), \chi_a(x)], Q\} + \frac{i}{m}\{\psi_b(y), [\chi_a(x), Q]\} \\
=& 0.
\end{aligned} \tag{5.2.8}$$

Repeating the same computation for $[\bar{\chi}_{\bar{b}}(x), \chi_a(y)]$, using $[Q, \bar{\chi}_{\bar{b}}(x)] = 0$, one gets:

$$[\bar{\chi}_{\bar{b}}(x), \chi_a(y)] = 0. \tag{5.2.9}$$

Since we want the ghost field to be a superfield as well we have $u = \frac{1}{2m}[Q_a, \chi^a]$ and therefore

$$\begin{aligned}
\{u(x), u(y)\} =& \frac{1}{2m}\{u(x), [Q_a, \chi^a(y)]\} \\
=& \frac{1}{2m}\big(\{Q_a, [\chi^a(y), u(x)]\} - [\chi^a(y), \{u(x), Q_a\}]\big) \\
=& 0,
\end{aligned} \tag{5.2.10}$$

because $\chi^a(y)$ commutes with $u(x)$ and $\{u(x), Q_a\} = 0$. In the same way one gets:

$$\begin{aligned}
\{u(x), u^K(y)\} =& -\frac{1}{2m}\{u(x), [\bar{Q}_{\bar{a}}, \bar{\chi}^{\bar{a}}(y)]\} \\
=& -\frac{1}{2m}\big(\{\bar{Q}_{\bar{a}}, [\bar{\chi}^{\bar{a}}(y), u(x)]\} - [\bar{\chi}^{\bar{a}}(y), \{u(x), \bar{Q}_{\bar{a}}\}]\big) \\
=& 0.
\end{aligned} \tag{5.2.11}$$

Therefore we are forced to introduce an anti-ghost field $\tilde{u}(x)$ if we want to quantize the ghost sector of the (unphysical) Hilbert space. We assume the following (anti)commutation relations:

$$\begin{aligned}
\{u_l(x), \tilde{u}_j(y)\} &= -i\delta_{ij} D(x-y); \\
[\chi_a(x), \tilde{\chi}_b(y)] &= im\epsilon_{ab} D(x-y); \\
[\bar{\chi}_{\bar{a}}(x), \tilde{\chi}_b(y)] &= \sigma^{\mu}_{b\bar{a}} \partial_\mu D(x-y).
\end{aligned} \tag{5.2.12}$$

They are consistent with the requirement that the anti-ghost fields are the components of a superfield too. First one notes that the first equation is consistent with $u_j^K(x) = u_j(x)$ and $\tilde{u}_l^K(x) = -\tilde{u}_l(x)$. Then if we assume that $i\{Q_a, \tilde{u}_1(x)\} = i\tilde{\chi}_a(x) = \{Q_a, \tilde{u}_2(x)\}$, , $[Q_a, \tilde{\chi}_b(x)] = m\epsilon_{ab}\tilde{u}(x)$ and $i[Q_a, \bar{\tilde{\chi}}_{\bar{b}}(x)] = -\sigma^{\mu}_{a\bar{b}}\partial_\mu \tilde{u}(x)$, then we get

$$\begin{aligned}
[\chi_a(x), \tilde{\chi}_b(y)] =& [\chi_a(x), \{Q_b, \tilde{u}_1(y)\}] \\
=& -\{Q_b, [\tilde{u}_1(y), \chi_a(x)]\} + \{\tilde{u}_1(y), [\chi_a(x), Q_b]\} \\
=& 0 + m\epsilon_{ba}\{\tilde{u}_1(y), u(x)\} \\
=& im\epsilon_{ab} D(x-y).
\end{aligned} \tag{5.2.13}$$

Applying the Dirac-equation one obtains $[\bar{\chi}_{\bar{a}}(x), \tilde{\chi}_b(y)] = \sigma^{\mu}_{b\bar{a}}\partial_\mu D(x-y)$.

5.3 Gauge Variation of The Anti-Ghost Field

Now that we have introduced this anti-ghost superfield, we have to investigate its gauge variation. Let us write

$$\begin{aligned}
u(x) &\stackrel{\text{def}}{=} u_1(x) + iu_2(x); \\
u^K(x) &\stackrel{\text{def}}{=} u_1(x) - iu_2(x); \\
\tilde{u}(x) &\stackrel{\text{def}}{=} \tilde{u}_1(x) + i\tilde{u}_2(x); \\
\tilde{u}^K(x) &\stackrel{\text{def}}{=} -\tilde{u}_1(x) + i\tilde{u}_2(x).
\end{aligned} \tag{5.3.1}$$

The equation

$$
\begin{aligned}
-2i\partial^y_\mu D(x-y) =&\{\tilde{u}^K(x), \partial^y_\mu u(y)\}\\
=&\{\tilde{u}^K(x), [Q, A_\mu(y)+iB_\mu(y)]\}\\
=&\big[\{Q,\tilde{u}^K(x)\}, A_\mu(y)+iB_\mu(y)\big] + \{Q, [A_\mu(y)+iB_\mu(y), \tilde{u}^K(x)]\}\\
=&\big[\{Q,\tilde{u}^K(x)\}, A_\mu(y)+iB_\mu(y)\big],
\end{aligned}
\tag{5.3.2}
$$

invoking the irreducibility of the field algebra, leads to

$$\{Q, \tilde{u}^K(x)\} = \partial^x_\nu\big(A^\nu(x) - iB^\nu(x)\big) - im\gamma^\dagger(x), \tag{5.3.3}$$

and consequently

$$\{Q, \tilde{u}(x)\} = \partial^x_\nu\big(A^\nu(x) + iB^\nu(x)\big) + im\gamma(x). \tag{5.3.4}$$

The field $\gamma(x)$ has to commute with all the other fields and must be the scalar component of some superfield. Therefore we must also introduce a spinor-field $g_a(x)$ and the relations

$$
\begin{aligned}
i[Q_a, \gamma(x)] =&0;\\
i[Q_a, \gamma^\dagger(x)] =&2g_a(x).
\end{aligned}
\tag{5.3.5}
$$

This field will be essential to assure the nilpotency of Q.

For the variation of $\tilde{\chi}_a(x)$ we need the already mentioned relation $\{Q_a, \tilde{u}^K(x)\} = 2\tilde{\chi}_a(x)$, which leads to

$$
\begin{aligned}
[Q, \tilde{\chi}_a(x)] =&\frac{-1}{2}\big[\{Q_a, \tilde{u}^K(x)\}, Q\big]\\
=&\frac{-1}{2}\big[\tilde{u}^K(x), \{Q, Q_a\}\big] - \frac{1}{2}\big[Q_a, \{\tilde{u}^K(x), Q\}\big]\\
=&\frac{-1}{2}\big[Q_a, \partial^x_\nu\big(A^\nu(x) - iB^\nu(x)\big) - im\gamma^\dagger(x)\big]\\
=&i\partial^x_\nu P^\nu_a(x) + mg_a(x).
\end{aligned}
\tag{5.3.6}
$$

Here we have used Assumption 1 and of course the super-algebraic (anti)commutation relations of the vector superfield.

To find out the gauge variation of $\tilde{\bar{\chi}}_a(x)$ one is tempted to take the conjugate of the last relation. However recall that $\tilde{\bar{\chi}}_a(x) = -\tilde{\chi}_a^K(x)$. We therefore get

$$[Q, \tilde{\bar{\chi}}_{\bar{a}}(x)] = -i\partial_\nu^x \bar{P}_{\bar{a}}^\nu(x) + m\bar{g}_{\bar{a}}(x). \tag{5.3.7}$$

This was the reason for the factor i in 5.3.4. Indeed, the gauge variation of $\tilde{\chi}_a(x)$ must be a field obeying the Majorana-equation, since, applied to $\tilde{\chi}_a(x)$, the Majorana-equation yields the anti-conjugate field $\tilde{\bar{\chi}}_{\bar{a}}(x)$.

The gauge variation of $\gamma(x)$ and $g_a(x)$ now follow easily:

$$\begin{aligned}
[Q, \gamma(x)] =& \frac{-i}{m}\big[Q, \{Q, \tilde{u}(x)\} - \partial_\nu\big(A^\nu(x) + iB^\nu(x)\big)\big] \\
=& \frac{i}{m}\partial_\nu\partial^\nu u(x) \\
=& -imu(x); \\
\{Q, g_a(x)\} =& \frac{1}{m}\{Q, [Q, \tilde{\chi}_a(x)] - i\partial_\mu P_a^\mu(x)\} \\
=& \frac{1}{m}\partial_\mu\partial^\mu \chi_a(x) \\
=& -m\chi_a(x).
\end{aligned} \tag{5.3.8}$$

This leads us to the equation

$$[Q, G(x, \theta, \bar{\theta})] = -imU(x, \theta, \bar{\theta}) \tag{5.3.9}$$

Now that we've worked out the (anti)commutation relations of the gauge charge with all the field components of the vector-, ghost-, antighost and goldstone-superfields, we can go over to the construction of Q.

5.4 Explicit Form of Q

From chapter 1 we know that a free Klein-Gordon field $\varphi(x)$ can be written as

$$\varphi(x) = \int_{y^0=x^0} d^3y D(x-y) \overleftrightarrow{\partial_{y^0}} \varphi(y). \tag{5.4.1}$$

Similarly for a Majorana field $\psi_a(x)$ one has:

$$\begin{aligned}
\psi_a(x) =& - im\sigma^0_{a\bar{b}} \int_{y^0=x^0} d^3yD(x-y)\bar{\psi}^{\bar{b}}(y) \\
&+\sigma^\mu_{a\bar{b}}\partial^x_\mu \int_{y^0=x^0} d^3yD(x-y)\bar{\sigma}^{0\bar{b}b}\psi_b(y); \\
\bar{\psi}^{\bar{a}}(x) =& - im\bar{\sigma}^{0\bar{a}b} \int_{y^0=x^0} d^3yD(x-y)\psi_b y \\
&+\bar{\sigma}^{\mu\bar{a}b}\partial^x_\mu \int_{y^0=x^0} d^3yD(x-y)\sigma^0_{b\bar{b}}\bar{\psi}^{\bar{b}}(y).
\end{aligned} \tag{5.4.2}$$

This together with equations 5.2.5 and 5.3.8 gives a term

$$Q(1) = \frac{-i}{2} \int d^3y(\partial_\nu c^{K\nu}(y) - im\gamma^\dagger(y)) \overleftrightarrow{\partial_{y^0}} u(y), \tag{5.4.3}$$

which must be a component of Q. One immediately sees that $\{Q, Q(1)\} = 0$.

A similar term has to be added for the spinor-fields. At the end of Appendix A, we show that the fields $i\partial P_a(x)$ and $-i\partial \bar{P}_{\bar{a}}(x)$ are both Majorana fields.

Together with assumption 1, equations 5.3.3, 5.3.4, 5.3.6 and 5.3.7 we arrive at the

Result 6. *The explicit form of the gauge charge Q is given by:*

$$\begin{aligned}
Q =& \frac{-i}{2} \int_{y^0=0} d^3y\big(\partial^y_\nu A^\nu(y) - i\partial^y_\nu B^\nu(y) - im\gamma^\dagger(y)\big) \overleftrightarrow{\partial_{y^0}} u(y) \\
&+\frac{i}{2} \int_{y^0=0} d^3y\big(\partial^y_\nu A^\nu(y) + i\partial^y_\nu B^\nu(y) + im\gamma(y)\big) \overleftrightarrow{\partial_{y^0}} u^K(y) \\
&+ \int_{y^0=0} d^3y\big(i\partial^y_\nu P^\nu_a(y) + mg_a(y)\big)\bar{\sigma}^{0\bar{b}a}\bar{\chi}_{\bar{b}}(y) \\
&+ \int_{y^0=0} d^3y\big(- i\partial^y_\nu \bar{P}^\nu_{\bar{a}}(y) + m\bar{g}_{\bar{a}}(y)\big)\bar{\sigma}^{0\bar{a}b}\chi_b(y).
\end{aligned} \tag{5.4.4}$$

From this expression one can read off the self-adjointness of Q. The (anti)commutation relation with the components of the various superfields give the expected results. This is of course not astonishing since we constructed Q just to meet this requirement. The nilpotency of Q follows from the computation of $\{Q, Q\}$:

$$\begin{aligned}\{Q,Q\} =&\frac{-i}{2}\int_{y^0=0} d^3y\big(m^2u^K(y) - m^2u^K(y)\big)\overset{\leftrightarrow}{\partial_{y^0}} u(y) + 0\\ &+\frac{i}{2}\int_{y^0=0} d^3y\big(-m^2u(y) + m^2u(y)\big)\overset{\leftrightarrow}{\partial_{y^0}} u^K(y) + 0\\ &+\int_{y^0=0} d^3y\big(m^2\chi_a(y) - m^2\chi_a(y)\big)\bar{\sigma}^{0\bar{b}a}\bar{\chi}_{\bar{b}}(y) + 0\\ &+\int_{y^0=0} d^3y\big(m^2\bar{\chi}_{\bar{a}}(y) - m^2\bar{\chi}_{\bar{a}}(y)\big)\bar{\sigma}^{0\bar{a}b}\chi_b(y) + 0\\ &=0,\end{aligned} \tag{5.4.5}$$

where we have used $\{Q, BF\} = [Q, B]F+B\{Q, F\}$, $\{Q, FB\} = -F[Q, B]+\{Q, F\}B$, where B (respectively F) is a bosonic (respectively fermionic) operator and the fact that the ghost super-field has a vanishing gauge variation. Therefore we see that $Q^2 = 0$ is fulfilled.

We can also factor out the unphysical components of the theory. Through the explicit form of the gauge-charge Q, one sees, that, as expected, all ghost-fields are unphysical, since they do not belong to the kernel of Q. Similarly the Goldstone-superfield $G(x, \theta, \bar{\theta})$ is clearly seen to be unphysical as are the scalar fields $\partial A(x)$, $\partial B(x)$ and the Majorana-field $\partial P_a(x)$. The latter fields constitute together a Ω_0-Supermultiplet as was seen at the end of chapter 4.

The remaining physical subspaces constitute therefore a Ω_1-Supermultiplet, consisting of one spin-$\frac{1}{2}$-multiplet, two spin-1-multiplets and a spin-$\frac{3}{2}$-multiplet.

Chapter 6

A Simplified Representation

6.1 Introduction

In chapter 4 we have seen that the Majorana field $\Psi_a(x)$ could actually be build in terms of the mixed field $P^a_\mu(x)$. The following question therefore arises: Is it possible to describe the same supersymmetric multiplet withoutmaking any reference to this Majorana field? The answer is a clear "yes" and furthermore, the commutation relations $[Q_a, P^a_\mu(x)]$ can be simplified.
This is precisely what we will attempt to do in this chapter.

6.2 New Fields from Old Ones

Because of the remark 4.4.1 we can claim that the fields $c_\mu(x)$, $c^K_\mu(x)$, $P^a_\mu(x)$ and $\bar{P}^{\bar{a}}_\mu(x)$ span the whole Hilbert space for the supersymmetric vector-multiplet. Therefore, we actually don't explicitly "need" the Majorana field $\Psi_a(x)$.
Furthermore the commutation relations 4.4.13 look very complex. But one can simplify them by introducing:

Definition 6.2.1. The fields $v_\mu(x)$ and $f_{\mu a}(x)$ are defined by

$$\begin{aligned} v_\mu(x) \stackrel{\text{def}}{=} & A c_\mu(x) + \frac{B}{m^2}\partial_\mu \partial c(x); \\ f_{\mu a}(x) \stackrel{\text{def}}{=} & C P_{\mu a}(x) + \frac{D}{m^2}\partial_\mu \partial P_a(x), \quad \mathcal{A}, B, C, D \in \mathbb{C}. \end{aligned} \tag{6.2.1}$$

Note that these fields are again solutions of the Klein-Gordon equation. The fields $c_\mu(x)$ and $P_{\mu a}(x)$ can therefore be recovered from the new defined ones

$$\begin{aligned} c_\mu(x) = & \frac{1}{A} v_\mu(x) - \frac{B}{A m^2}\partial_\mu \partial c(x) \\ = & \frac{1}{A} v_\mu(x) - \frac{B}{A(A-B)m^2}\partial_\mu \partial v(x) \\ P_{\mu a}(x) = & \frac{1}{C} f_{\mu a}(x) - \frac{D}{C(C-D)m^2}\partial_\mu \partial f_a(x), \end{aligned} \tag{6.2.2}$$

provided that $A \neq B$ and $C \neq D$. In this case, no "information" is lost through this transformation of the fields.

These new fields have now the following (anti)commutation relations with the superalgebra:

$$\begin{aligned} i[Q_a, v_\mu(x)] = & 0; \\ i[Q_a, v^K_\mu(x)] = & i[Q_a, \bar{A} c^K_\mu(x) + \frac{\bar{B}}{m^2}\partial_\mu \partial c^K(x)] \\ = & 2\bar{A} P_{\mu a}(x) + \frac{2\bar{B}}{m^2}\partial_\mu \partial P_a(x); \\ i\{Q_a, f_{\mu b}(x)\} = & i\{Q_a, C P_{\mu a}(x) + \frac{D}{m^2}\partial_\mu \partial P_a(x)\} \\ = & m\epsilon_{ab}\big(C c_\mu(x) + \frac{2C-D}{m^2}\partial_\mu \partial c(x)\big) \\ i\{Q_a, \bar{f}_{\mu \bar{b}}(x)\} = & i\sigma^\nu_{a\bar{b}}\partial_\nu\big(\bar{C} c_\mu(x) + \frac{\bar{D}}{m^2}\partial_\mu \partial c(x)\big). \end{aligned} \tag{6.2.3}$$

If we chose the constants such that $A = C = \bar{A} = \bar{C}$, $2C - D = B$ and $\bar{B} = D$ this simply gives

$$\begin{aligned} i[Q_a, v_\mu(x)] =& 0; \\ i[Q_a, v^K_\mu(x)] =& 2 f_{\mu a}(x); \\ i\{Q_a, f_{\mu b}(x)\} =& m\epsilon_{ab} v_\mu(x) \\ i\{Q_a, \bar{f}_{\mu \bar{b}}(x)\} =& i\sigma^\nu_{a\bar{b}} \partial_\nu v_\mu(x). \end{aligned} \tag{6.2.4}$$

Further constraints on the constants arise from the commutation relations:

$$\begin{aligned} [v_\mu(x), v^K_\nu(y)] =& 2i\eta_{\mu\nu} D_m(x-y) \\ =& (A\delta^\alpha_\mu + \frac{B}{m^2}\partial_\mu\partial^\alpha)(\bar{A}\delta^\beta_\nu + \frac{\bar{B}}{m^2}\partial^y_\nu\partial^\beta_y)[c_\alpha(x), c_\beta(y)] \\ =& 2i(A\delta^\alpha_\mu + \frac{B}{m^2}\partial_\mu\partial^\alpha)(\bar{A}\delta^\beta_\nu + \frac{\bar{B}}{m^2}\partial_\nu\partial^\beta)\eta_{\alpha\beta} D_m(x-y) \\ =& 2i(|A|^2\eta_{\mu\nu} + (\frac{A\bar{B}}{m^2} + \frac{\bar{A}B}{m^2} - \frac{|B|^2}{m^2})\partial_\mu\partial_\nu) D_m(x-y), \\ \Rightarrow& |A|^2 = 1, \quad \bar{B}(A-B) + B\bar{A} = 0. \end{aligned} \tag{6.2.5}$$

Without loss of generality one can chose $A = 1 = C$ and $B = 1 + i = \bar{D}$.

The anticommutation relations for the mixed field also simplify considerably:

$$\begin{aligned} \{f_{\mu a}(x), f_{\nu b}(y)\} =& -im\epsilon_{ab}\eta_{\mu\nu} D_m(x-y), \\ \{f_{\mu a}(x), \bar{f}_{\nu \bar{b}}(y)\} =& -\sigma^\alpha_{a\bar{b}}\partial_\alpha \eta_{\mu\nu} D_m(x-y). \end{aligned} \tag{6.2.6}$$

6.3 Gauge Variation of The New Fields

Using the gauge-variation of our vector superfield defined in chapter 4, one can easily compute the gauge variation of the new one:

$$
\begin{aligned}
[Q, v_\mu(x)] =& [Q, c_\mu(x) + \frac{1+i}{m^2}\partial_\mu \partial c(x)] \\
=& \partial_\mu u(x) - \partial_\mu u(x) - i\partial_\mu u(x) \\
=& - i\partial_\mu u(x); \\
[Q, v^K_\mu(x)] =& - i\partial_\mu u^K(x); \\
\{Q, f_{\mu a}(x)\} =& \{Q, P_{\mu a}(x) + \frac{1-i}{m^2}\partial_\mu \partial P_a(x)\} \\
=& i\partial_\mu \chi_a(x) - i\partial_\mu \chi_a(x) - \partial_\mu \chi_a(x) \\
=& - \partial_\mu \chi_a(x); \\
\{Q, \bar{f}_{\mu \bar{a}}(x)\} =& - \partial_\mu \bar{\chi}_{\bar{a}}(x).
\end{aligned}
\tag{6.3.1}
$$

Up to a constant, these are the same relations as the one obtained in chapter 5. Therefore, the transformation 6.2.1 leaves the physical and unphysical subspaces of our Fock-space unchanged.

Appendix A

Lie Series

A.1 Introduction

The suoerfields where up to now computed with the help of the Baker-Hausdorff formula. Equivalently these fields can be computed using the Lie-series expansion for the exponential:

$$\begin{aligned} exp(iB)\varphi_A(x)exp(-iB) =& \sum_{j=0}^{\infty} \frac{1}{j!} C_j(B, \varphi_A(x))), ; \\ C_0(B, \varphi_A(x))) =& \varphi_A(x), \\ C_j(B, \varphi_A(x)) =& [iB, C_{j-1}(B, \varphi_A(x))], \quad \forall j \geq 1. \end{aligned} \tag{A.1.1}$$

We show in this appendix that both methods coincide.

A.2 The Scalar Chiral Superfield

In this case $\varphi_A(x) = \varphi(x)$ and $iB = i\theta Q + i\bar{\theta}\bar{Q} \overset{\text{def}}{=} iS$. Definition 4.2.2 gives:

$$\begin{aligned}
C_0(S, \varphi(x)) =& \varphi(x);\\
C_1(S, \varphi(x)) =& 2\bar{\theta}\bar{\psi}(x);\\
C_2(S, \varphi(x)) =& 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) - 2m\bar{\theta}\bar{\theta}\varphi^\dagger(x);\\
C_3(S, \varphi(x)) =& 4i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\theta}\bar{\psi}(x) - 4m\bar{\theta}\bar{\theta}\theta\psi(x);\\
C_4(S, \varphi(x)) =& 6m^2\theta\theta\bar{\theta}\bar{\theta}\varphi(x);\\
C_n(S, \varphi(x)) =& 0, \quad \forall n \geq 5.
\end{aligned} \tag{A.2.1}$$

Summing up we arrive at the expression

$$\begin{aligned}
\Phi(\theta, \bar{\theta}, x) =& \sum_{k=0}^{4} \frac{1}{k!} C_k(S, \varphi(x))\\
=& \varphi(x) + 2\bar{\theta}\bar{\psi}(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) - m\bar{\theta}\bar{\theta}\varphi^\dagger(x)\\
&+ \frac{2}{3} i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\theta}\bar{\psi}(x) - \frac{2}{3} m\bar{\theta}\bar{\theta}\theta\psi(x) + \frac{1}{4} m^2\theta\theta\bar{\theta}\bar{\theta}\varphi(x)\\
=& (1 + i\theta\sigma^\mu\bar{\theta}\partial_\mu + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta})\varphi(x)\\
&+ 2\bar{\theta}\bar{\psi}(x) - m\bar{\theta}\bar{\theta}\theta\psi(x)\\
&- m\bar{\theta}\bar{\theta}\varphi^\dagger(x)\\
=& 4.2.16,
\end{aligned} \tag{A.2.2}$$

where we have used that

$$
\begin{aligned}
2i\theta\sigma^\mu\bar\theta\partial_\mu\bar\theta\bar\psi(x) =&2i\theta^a\big(\sigma^\mu_{a\bar1}\bar\theta^{\bar1}+\sigma^\mu_{a\bar2}\bar\theta^{\bar2}\big)\partial_\mu\bar\theta\bar\psi(x)\\
=&2i\theta^a\big(\sigma^\mu_{a\bar1}\bar\theta^{\bar1}\partial_\mu\bar\theta_{\bar1}\bar\psi^{\bar1}(x)+\sigma^\mu_{a\bar2}\bar\theta^{\bar2}\partial_\mu\bar\theta_{\bar2}\bar\psi^{\bar2}(x)\big)\\
=&-2i\theta^a\big(\sigma^\mu_{a\bar1}\partial_\mu\bar\psi^{\bar1}(x)\bar\theta^{\bar1}\bar\theta_{\bar1}+\sigma^\mu_{a\bar2}\partial_\mu\bar\psi^{\bar2}(x)\bar\theta^{\bar2}\bar\theta_{\bar2}\big)\\
=&-i\theta^a\big(\sigma^\mu_{a\bar1}\partial_\mu\bar\psi^{\bar1}(x)(\bar\theta^{\bar1}\bar\theta_{\bar1}+\bar\theta^{\bar1}\bar\theta_{\bar1})+\sigma^\mu_{a\bar2}\partial_\mu\bar\psi^{\bar2}(x)(\bar\theta^{\bar2}\bar\theta_{\bar2}+\bar\theta^{\bar2}\bar\theta_{\bar2})\big)\\
=&-i\theta^a\big(\sigma^\mu_{a\bar1}\partial_\mu\bar\psi^{\bar1}(x)(\bar\theta^{\bar1}\bar\theta_{\bar1}-\bar\theta_{\bar2}\bar\theta^{\bar2})+\sigma^\mu_{a\bar2}\partial_\mu\bar\psi^{\bar2}(x)(\bar\theta^{\bar2}\bar\theta_{\bar2}-\bar\theta_{\bar1}\bar\theta^{\bar1})\big)\\
=&-i\theta\sigma^\mu\partial_\mu\bar\psi(x)\bar\theta\bar\theta=-m\bar\theta\bar\theta\theta\psi(x).
\end{aligned}
\tag{A.2.3}
$$

A.3 The Chiral Ghost Superfield

To compare the Lie-Series method with the Hausdorff-formula when starting with the spinor component of the super-field, which we shall do now, we will need the following equality:

$$
\sigma^\mu_{a\bar b}\partial_\mu\chi_c(x)=\sigma^\mu_{c\bar b}\partial_\mu\chi_a(x)-im\epsilon_{ac}\bar\chi_{\bar b}(x).
\tag{A.3.1}
$$

Now, the Lie-series give:

$$
\begin{aligned}
C_0(S,\chi_a(x)) =&\chi_a(x);\\
C_1(S,\chi_a(x)) =&im\theta_a u(x)-\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\partial_\mu u^K(x);\\
C_2(S,\chi_a(x)) =&-2m\theta_a\bar\theta\bar\chi(x)-2i\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\partial_\mu\theta\chi(x)\\
\overset{A.3.1}{=}&-4i\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\partial_\mu\theta\chi(x)-2i\theta\sigma^\mu\bar\theta\partial_\mu\chi_a(x);
\end{aligned}
\tag{A.3.2}
$$

$$
\begin{aligned}
C_3(S,\chi_a(x)) =& 4m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta u(x) + 4i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\sigma^\nu\bar{\theta}\partial_\nu u^K(x)\\
&+2m\theta\sigma^\mu\bar{\theta}\partial_\mu\theta_a u(x) + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\nu u^K(x)\\
=& 6m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta u(x) + 6i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\sigma^\nu\bar{\theta}\partial_\nu u^K(x)\\
&+2m\theta\sigma^\mu\bar{\theta}\partial_\mu\theta_a u(x) - 2m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta u(x)\\
=& 6m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta u(x) + 6i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\sigma^\nu\bar{\theta}\partial_\nu u^K(x)\\
&+6m\theta\sigma^\mu\bar{\theta}\partial_\mu\theta_a u(x); \qquad \text{(A.3.3)}\\
C_4(S,\chi_a(x)) =& 12im\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta\bar{\theta}\bar{\chi}(x) - 12\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\sigma^\nu\bar{\theta}\partial_\nu\theta\chi(x)\\
&+12im\theta\sigma^\mu\bar{\theta}\partial_\mu\theta_a\bar{\theta}\bar{\chi}(x)\\
=& -6m^2\bar{\theta}\bar{\theta}\theta\theta\chi_a(x) - 3m^2\bar{\theta}\bar{\theta}\theta\theta\chi_a(x)\\
&+3m^2\bar{\theta}\bar{\theta}\theta\theta\chi_a(x)\\
=& -6m^2\bar{\theta}\bar{\theta}\theta\theta\chi_a(x)
\end{aligned}
$$

Putting all together we arrive at

$$
\begin{aligned}
X_a(x,\theta,\bar{\theta}) =& \chi_a(x) + im\theta_a u(x) - \sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu u^K(x)\\
&-2i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\chi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\chi_a(x)\\
&+m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\theta u(x) + i\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\mu\theta\sigma^\nu\bar{\theta}\partial_\nu u^K(x) \qquad \text{(A.3.4)}\\
&+m\theta\sigma^\mu\bar{\theta}\partial_\mu\theta_a u(x)\\
&-\frac{m^2}{4}\bar{\theta}\bar{\theta}\theta\theta\chi_a(x)
\end{aligned}
$$

This has to be compared with the expression in result 2:

$$
\begin{aligned}
X_a(x,\theta,\bar{\theta}) \overset{\text{def}}{=}& W(\theta,\bar{\theta})\chi_a(x)W^{-1}(\theta,\bar{\theta})\\
=& \big(1 - i\theta\sigma^\mu\bar{\theta}\partial_\mu + \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\big)\big(\chi_a(x) + im\theta_a u(x) \qquad \text{(A.3.5)}\\
&-\sigma^\mu_{a\bar{a}}\bar{\theta}^{\bar{a}}\partial_\mu(u^K(x) + 2i\theta\chi(x) - m\theta\theta u(x))\big).
\end{aligned}
$$

Every term agrees explicitly up to the terms containing four Grassman variables. For those note that

$$
\begin{aligned}
&\frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) - 2\theta\sigma^\mu\bar{\theta}\partial_\mu\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\partial_\nu\theta\chi(x) \\
&= \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) - 2\theta^c\theta^d\sigma^\mu_{c\bar{a}}\bar{\theta}^{\bar{a}}\bar{\theta}^{\bar{b}}\partial_\mu\sigma^\nu_{a\bar{b}}\partial_\nu\chi_d(x) \\
&= \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) - \frac{1}{2}\theta\theta\epsilon^{dc}\sigma^\mu_{c\bar{a}}\bar{\theta}\bar{\theta}\epsilon^{\bar{a}\bar{b}}\partial_\mu\sigma^\nu_{a\bar{b}}\partial_\nu\chi_d(x) \\
&= \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) - \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\epsilon^{dc}(\sigma^\mu\bar{\sigma}^\nu)_{ca}\partial_\mu\partial_\nu\chi_d(x) \\
&= \frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) - \frac{m^2}{2}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x) \\
&= \frac{-m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\chi_a(x),
\end{aligned}
\tag{A.3.6}
$$

which is the desired result.

A.4 The Vector Superfield

Here we take $\varphi_A(x) = \psi_a(x)$. Using 4.4.19 we arrive at:

$$
\begin{aligned}
C_0(S,\varphi(x)) =& \psi_a(x);\\
C_1(S,\varphi(x)) =& i(\sigma^\nu\bar\sigma^\mu)_{ab}\theta^b\partial_\nu c_\mu(x) + m\sigma^\mu_{a\bar b}\bar\theta^{\bar b}c^K_\mu(x);\\
C_2(S,\varphi(x)) =& 2i(\sigma^\nu\bar\sigma^\mu)_{ab}\theta^b\partial_\nu\bar\theta\bar P_\mu(x) + 2m\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta P_\mu(x);\\
C_3(S,\varphi(x)) =& 2i(\sigma^\nu\bar\sigma^\mu)_{ab}\theta^b\partial_\nu\big(i\theta\sigma^\alpha\bar\theta\partial_\alpha c_\mu(x) - m\bar\theta\bar\theta\big(c^K_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c^K(x)\big)\big)\\
&-2m\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\big(i\theta\sigma^\alpha\bar\theta\partial_\alpha c^K_\mu(x) + m\theta\theta\big(c_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c(x)\big)\big)\\
=&-2\sigma^\nu_{a\bar b}\bar\sigma^{\mu\bar b c}\epsilon_{bc}\theta^b\partial_\nu\theta^d\sigma^\alpha_{d\bar c}\bar\theta^{\bar c}\partial_\alpha c_\mu(x)\\
&-2i(\sigma^\nu\bar\sigma^\mu)_{ab}\theta^b\partial_\nu m\bar\theta\bar\theta\big(c^K_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c^K(x)\big)\\
&-2im\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta^c\sigma^\alpha_{c\bar a}\bar\theta^{\bar a}\partial_\alpha c^K_\mu(x)\\
&-2m^2\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta\theta\big(c_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c(x)\big);\\
=&-(\sigma^\nu\bar\sigma^\mu\sigma^\alpha)_{a\bar c}\theta\theta\bar\theta^{\bar c}\partial_\nu\partial_\alpha c_\mu(x) \qquad \text{(A.4.1)}\\
&-2i(\sigma^\nu\bar\sigma^\mu)_{ab}\theta^b\partial_\nu m\bar\theta\bar\theta c^K_\mu(x)\\
&-\frac{2i}{m}(\sigma^{(\nu}\bar\sigma^{\mu)})_{ab}\theta^b\partial_\nu\bar\theta\bar\theta\partial_\mu\partial c^K(x)\big)\\
&+im(\sigma^\mu\bar\sigma^\alpha)_{ab}\bar\theta\bar\theta\theta^b\partial_\alpha c^K_\mu(x)\\
&-2m^2\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta\theta\big(c_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c(x)\big)\\
=&-(i\epsilon^{\alpha\mu\nu}_\lambda\sigma^\lambda - g^{\alpha\nu}\sigma^\mu + g^{\alpha\mu}\sigma^\nu + g^{\mu\nu}\sigma^\alpha)_{a\bar c}\theta\theta\bar\theta^{\bar c}\partial_\nu\partial_\alpha c_\mu(x)\\
&+2i(2g^{\mu\nu}\epsilon_{ab} + (\sigma^\mu\bar\sigma^\nu)_{ab})\theta^b\partial_\nu m\bar\theta\bar\theta c^K_\mu(x)\\
&+\frac{4i}{m}g^{\mu\nu}\theta_a\partial_\nu\bar\theta\bar\theta\partial_\mu\partial c^K(x)\big)\\
&+im(\sigma^\mu\bar\sigma^\alpha)_{ab}\bar\theta\bar\theta\theta^b\partial_\alpha c^K_\mu(x)\\
&-2m^2\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta\theta\big(c_\mu(x) + \frac{2}{m^2}\partial_\mu\partial c(x)\big)
\end{aligned}
$$

$$
\begin{aligned}
&=(\Box\sigma^\mu - 2\partial^\mu\sigma^\nu\partial_\nu)_{a\bar{c}}\theta\theta\bar{\theta}^{\bar{c}}c_\mu(x) \\
&\quad +3i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\partial_\nu m\bar{\theta}\bar{\theta}c^K_\mu(x) \\
&\quad -2m^2\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta\big(c_\mu(x)+\frac{2}{m^2}\partial_\mu\partial c(x)\big) \\
&= +3i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\partial_\nu m\bar{\theta}\bar{\theta}c^K_\mu(x) \\
&\quad -3m^2\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta\big(c_\mu(x)+\frac{2}{m^2}\partial_\mu\partial c(x)\big); \\
C_4(S,\varphi(x)) &= +6i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\partial_\nu m\bar{\theta}\bar{\theta}\theta P_\mu(x) \\
&\quad -6m^2\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta\big(\bar{\theta}\bar{P}_\mu(x)+\frac{2}{m^2}\partial_\mu\partial\bar{\theta}\bar{P}(x)\big); \\
C_n(S,\varphi(x)) &=0, \quad \forall n\geq 5.
\end{aligned} \tag{A.4.2}
$$

Finally this gives the expression

$$
\begin{aligned}
V_a(x,\theta,\bar{\theta}) =&\psi_a(x) \\
&+i(\sigma^\nu\bar{\sigma}^\mu)_{ab}\theta^b\partial_\nu c_\mu(x) + m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}c^K_\mu(x) \\
&+i(\sigma^\nu\bar{\sigma}^\mu)_{ab}\theta^b\partial_\nu\bar{\theta}\bar{P}_\mu(x) + m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta P_\mu(x) \\
&+\frac{1}{2}i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\partial_\nu m\bar{\theta}\bar{\theta}c^K_\mu(x) \\
&-\frac{1}{2}m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta\big(c_\mu(x)+\frac{2}{m^2}\partial_\mu\partial c(x)\big) \\
&+\frac{1}{4}i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\partial_\nu m\bar{\theta}\bar{\theta}\theta P_\mu(x) \\
&-\frac{1}{4}m^2\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta\big(\bar{\theta}\bar{P}_\mu(x)+\frac{2}{m^2}\partial_\mu\partial\bar{\theta}\bar{P}(x)\big).
\end{aligned} \tag{A.4.3}
$$

This has to be compared with the equation 4.4.20, which we rewrite here by expliciting its polynomial character in the Grassmanian variables θ and $\bar{\theta}$:

$$\begin{aligned} V_a(x,\theta,\bar{\theta}) =&\psi_a(x) \\ &+i\theta^b(\sigma^\nu\bar{\sigma}^\mu)_{ab}\partial_\nu c_\mu(x) + m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}} c^K_\mu(x) \\ &-i\theta\sigma^\mu\bar{\theta}\partial_\mu\psi_a(x) + 2m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta P_\mu(x) \\ &+\theta\sigma^\alpha\bar{\theta}\partial_\alpha\theta^b(\sigma^\nu\bar{\sigma}^\mu)_{ab}\partial_\nu c_\mu(x) - m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x) \\ &-im\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\sigma^\alpha\bar{\theta}\partial_\alpha c^K_\mu(x) \\ &+\frac{m^2}{4}\theta\theta\bar{\theta}\bar{\theta}\psi_a(x) - 2im\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\sigma^\alpha\bar{\theta}\partial_\alpha\theta P_\mu(x). \end{aligned} \tag{A.4.4}$$

At first glance these two expression seem not to agree. Only the three first terms are explicitly equal.

However, for the quadratical terms in the variables θ and $\bar{\theta}$ one has:

$$\begin{aligned} &i(\sigma^\nu\bar{\sigma}^\mu)_{ab}\theta^b\partial_\nu\bar{\theta}\bar{P}_\mu(x) + m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta P_\mu(x) \\ \overset{4.4.11}{=} &-i\theta\sigma^\mu\bar{\theta}\partial_\mu\psi_a(x) + 2m\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta P_\mu(x). \end{aligned} \tag{A.4.5}$$

For the terms containing three Grassman variables note that:

$$\begin{aligned} &\theta\sigma^\alpha\bar{\theta}\partial_\alpha\theta^b(\sigma^\nu\bar{\sigma}^\mu)_{ab}\partial_\nu c_\mu(x) - m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x) \\ &-im\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\sigma^\alpha\bar{\theta}\partial_\alpha c^K_\mu(x) \\ &=\theta^c\sigma^\alpha_{c\bar{a}}\bar{\theta}^{\bar{a}}\partial_\alpha\theta^b\sigma^\nu_{a\bar{b}}\bar{\sigma}^{\mu\bar{b}d}\epsilon_{bd}\partial_\nu c_\mu(x) - m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x) \\ &-im\sigma^\mu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta^b\sigma^\alpha_{b\bar{c}}\bar{\theta}^{\bar{c}}\partial_\alpha c^K_\mu(x) \\ &=\frac{\bar{\theta}^{\bar{a}}}{2}\theta\theta(\sigma^\nu\bar{\sigma}^\mu\sigma^\alpha)_{a\bar{a}}\partial_\alpha\partial_\nu c_\mu(x) - m^2\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x) \\ &+\frac{im\theta^b}{2}\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\alpha)_{ab}\partial_\alpha c^K_\mu(x) \\ =&-\frac{m^2}{2}\sigma^\nu_{a\bar{b}}\bar{\theta}^{\bar{b}}\theta\theta(\delta^\mu_\nu + \frac{2}{m^2}\partial^{2\mu}_\nu)c_\mu(x) \\ &+\frac{im\theta^b}{2}\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\alpha)_{ab}\partial_\alpha c^K_\mu(x), \end{aligned} \tag{A.4.6}$$

showing that both agree.

For the terms containing four Grassman variables we will need two equations, which both derive from 4.4.14:

$$\begin{aligned} \sigma^{\mu}_{a\bar{b}}\partial_\mu\partial\bar{P}^{\bar{b}}(x) =& \frac{im}{2}\sigma^{\mu}_{a\bar{b}}\partial_\mu\epsilon^{\bar{b}\bar{a}}\big(\bar{\psi}_{\bar{a}}(x)+\sigma^{\nu}_{c\bar{a}}P^{c}_{\nu}(x)\big) \\ =& \frac{m^2}{2}\psi_a(x)+\frac{im}{2}(\sigma^\mu\bar{\sigma}^\nu)_{ac}\partial_\mu P^c_\nu(x); \\ \sigma^{\mu}_{a\bar{b}}\bar{P}^{\bar{b}}_{\mu} =& \frac{2i}{m}\partial P_a(x)-\psi_a(x). \end{aligned} \tag{A.4.7}$$

Consider now the terms in A.4.3 containing four Grassman variables. They reed:

$$\begin{aligned} &\frac{1}{4}i(\sigma^\mu\bar{\sigma}^\nu)_{ab}\theta^b\theta^c\partial_\nu m\bar{\theta}\bar{\theta}P_{\mu c}(x) \\ &-\frac{1}{4}m^2\sigma^{\mu}_{a\bar{b}}\bar{\theta}^{\bar{b}}\bar{\theta}_{\bar{c}}\theta\theta\big(\bar{P}^{\bar{c}}_{\mu}(x)+\frac{2}{m^2}\partial_\mu\partial\bar{P}^{\bar{c}}(x)\big) \\ =&\ \frac{im}{8}\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\nu)^{c}_{a}\partial_\nu P_{\mu c}(x) \\ &+\frac{m^2}{8}\theta\theta\bar{\theta}\bar{\theta}\sigma^{\mu}_{a\bar{b}}\big(\bar{P}^{\bar{b}}_{\mu}(x)+\frac{2}{m^2}\partial_\mu\partial\bar{P}^{\bar{b}}\big) \\ =&\ \frac{im}{8}\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\nu)^{c}_{a}\partial_\nu P_{\mu c}(x) \\ &+\frac{m^2}{8}\theta\theta\bar{\theta}\bar{\theta}\big(\frac{2i}{m}\partial P_a(x)-\psi_a(x)+\psi_a(x)-\frac{i}{m}(\sigma^\mu\bar{\sigma}^\nu)_{a}{}^{c}\partial_\mu P_{\nu c}(x)\big) \\ =&\ \frac{im}{8}\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\nu)^{c}_{a}\partial_\nu P_{\mu c}(x) \\ &+\frac{im}{8}\theta\theta\bar{\theta}\bar{\theta}\big(2\partial P_a(x)-2\partial P_a(x)+(\sigma^\nu\bar{\sigma}^\mu)_{a}{}^{c}\partial_\mu P_{\nu c}(x)\big) \\ =&\ \frac{im}{4}\theta\theta\bar{\theta}\bar{\theta}(\sigma^\nu\bar{\sigma}^\mu)_{a}{}^{c}\partial_\mu P_{\nu c}(x). \end{aligned} \tag{A.4.8}$$

Now we compute

$$
\begin{aligned}
m(\sigma^\mu\bar\sigma^\alpha)_b{}^c\partial_\alpha P_{\mu c}(x) \overset{4.4.11}{=} & i(\sigma^\nu\bar\sigma^\mu\sigma^\alpha)_{b\bar a}\partial^2_{\alpha\nu}\bar P^{\bar a}_\mu(x) - i(\sigma^\mu\bar\sigma^\alpha)_c{}^c\partial^2_{\mu\alpha}\psi_b(x)\\
=&2im^2\psi_b(x) + i(-g^{\alpha\nu}\sigma^\mu + g^{\alpha\mu}\sigma^\nu + g^{\nu\mu}\sigma^\alpha)_{b\bar a}\partial^2_{\alpha\nu}\bar P^{\bar a}_\mu(x)\\
=&2im^2\psi_b(x) + im^2\sigma^\mu_{b\bar a}\bar P^{\bar a}_\mu + 2i\sigma^\nu_{b\bar a}\partial_\nu\partial\bar P^{\bar a}(x)\\
\overset{A.4.7}{=}&2im^2\psi_b(x) - 2m^2\partial P_a(x) - im^2\psi_b(x) + im^2\psi_b(x) + m(\sigma^\mu\bar\sigma^\nu)_b{}^c\partial_\mu P_{\nu c}(x)\\
=&2im^2\psi_b(x) - m(\sigma^\nu\bar\sigma^\mu)_b{}^c\partial_\mu P_{\nu c}(x)\\
\Rightarrow m(\sigma^\mu\bar\sigma^\alpha)_b{}^c\partial_\alpha P_{\mu c}(x) =&im^2\psi_b(x).
\end{aligned}
\tag{A.4.9}
$$

As a consequence the expression in A.4.4 gives:

$$
\begin{aligned}
&\frac{m^2}{4}\theta\theta\bar\theta\bar\theta\psi_a(x) - 2im\sigma^\mu_{a\bar b}\bar\theta^{\bar b}\theta\sigma^\alpha\bar\theta\partial_\alpha\theta P_\mu(x)\\
=&\frac{m^2}{4}\theta\theta\bar\theta\bar\theta\psi_a(x) + \frac{im}{2}\bar\theta\bar\theta\theta\theta(\sigma^\mu\bar\sigma^\alpha)_a{}^c\partial_\alpha P_{\mu c}(x)\\
=&\frac{im}{4}\bar\theta\bar\theta\theta\theta(\sigma^\mu\bar\sigma^\alpha)_a{}^c\partial_\alpha P_{\mu c}(x),
\end{aligned}
\tag{A.4.10}
$$

which shows that both computations agree.

Remark A.4.1. From the previous computations one can show that the field $i\partial\bar P^{\bar b}$ satisfies the Majorana-equation. Indeed, from A.4.7 we get:

$$
\begin{aligned}
\sigma^\mu_{a\bar b}\partial_\mu\partial\bar P^{\bar b}(x) =&\frac{im}{2}\sigma^\mu_{a\bar b}\partial_\mu\epsilon^{\bar b\bar a}\left(\bar\psi_{\bar a}(x) + \sigma^\nu_{c\bar a}P^c_\nu(x)\right)\\
=&\frac{m^2}{2}\psi_a(x) + \frac{im}{2}(\sigma^\mu\bar\sigma^\nu)_{ac}\partial_\mu P^c_\nu(x)\\
=&\frac{m^2}{2}\psi_a(x) + \frac{im}{2}\left(2g^{\mu\nu}\epsilon_{ca} - (\sigma^\mu\bar\sigma^\nu)_{ac}\right)\partial_\mu P^c_\nu(x)\\
=&\frac{m^2}{2}\psi_a(x) - im\partial P_a(x) + (\sigma^\mu\bar\sigma^\nu)_a{}^c\partial_\mu P_{\nu c}(x)\\
\overset{A.4.9}{=}& - im\partial P_a(x);\\
\Rightarrow i\sigma^\mu_{a\bar b}\partial_\mu\big(i\partial\bar P^{\bar b}(x)\big) =&m\big(i\partial P_a(x)\big).
\end{aligned}
\tag{A.4.11}
$$

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